FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 29

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CONTENTS

Last day: One last bit of rational maps. Curves.

Today: A bit more curves. Introduction to cohomology.

1. SCHEME-THEORETIC CLOSURE, AND SCHEME-THEORETIC IMAGE

I discussed the scheme-theoretic closure of a locally closed scheme, and more generally, the scheme-theoretic image of a morphism. I've moved this discussion into the class 27 notes.

2. CURVES

Last day we proved a couple of important theorems:

2.1. *Key Proposition. — Suppose C is a dimension 1 finite type k-scheme, and p is a nonsingular point of it. Suppose Y is a projective k-scheme. Then any morphism $C - p \to Y$ extends to $C \to Y$.*

2.2. *Theorem. — If C is a nonsingular curve, then there is some projective nonsingular curve C' and an open immersion $C \hookrightarrow C'$.*

2.3. *Theorem. — The following categories are equivalent.*

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(i) *nonsingular projective curves, and surjective morphisms.*
(ii) *nonsingular projective curves, and dominant morphisms.*
(iii) *nonsingular projective curves, and dominant rational maps*
(iv) *quasiprojective reduced curves, and dominant rational maps*
(v) *fields of transcendence dimension* 1 *over* k, *and* k*-homomorphisms.*

We then discussed the degree of a morphism between projective nonsingular curves. In particular, we are in the midst of showing that any non-constant morphism from one projective nonsingular curve to another has a well-behaved degree. Suppose $f : C \to C'$ is a surjective (or equivalently, dominant) map of nonsingular projective curves. We showed that f is a finite morphism, by showing that f is the normalization of C' in the function field of C; hence the result follows by finiteness of integral closure.

2.4. *Proposition. — Suppose that* $\pi : C \to C'$ *is a surjective finite morphism, where* C *is an integral curve, and* C' *is an integral nonsingular curve. Then* $\pi_* \mathcal{O}_C$ *is locally free of finite rank.*

All we will really need is that C is reduced of pure dimension 1.

We are about to prove this.

Let's discuss again what this means. (I largely said this last day.) Suppose d is the rank of this allegedly locally free sheaf. Then the fiber over any point of C with residue field K is the Spec of an algebra of dimension d over K. This means that the number of points in the fiber, counted with appropriate multiplicity, is always d.

Proof. (For experts: we will later see that what matters here is that the morphism is finite and *flat*. But we don't yet know what flat is.)

The question is local on the target, so we may assume that C' is affine. Note that $\pi_* \mathcal{O}_C$ is torsion-free (as $\Gamma(C, \mathcal{O}_C)$ is an integral domain). Our plan is as follows: by an important exercise from last quarter (Exercise 5.2 of class 15; problem 10 on problem set 7), if the rank of the coherent sheaf $\pi_* \mathcal{O}_C$ is constant, then (as C' is reduced) $\pi_* \mathcal{O}_C$ is locally free. We'll show this by showing the rank at any closed point of C' is the same as the rank at the generic point.

The notion of "rank at a point" behaves well under base change, so we base change to the discrete valuation ring $\mathcal{O}_{C',p}$, where p is some closed point of C'. Then $\pi_* \mathcal{O}_C$ is a finitely generated module over a discrete valuation ring which is torsion-free. By the classification of finitely generated modules over a principal ideal domain, any finitely generate module over a principal ideal domain A is a direct sum of modules of the form $A/(d)$ for various $d \in A$. But if A is a discrete valuation ring, and $A/(d)$ is torsion-free, then $A/(d)$ is necessarily A (as for example all ideals of A are of the form 0 or a power of the maximal ideal). Thus we are done. $\square$

Remark. If we are working with complex curves, this notion of degree is the same as the notion of the topological degree.

Suppose C is a projective curve, and $\mathcal{L}$ is an invertible sheaf. We will define $\deg \mathcal{L}$.

Let s be a non-zero rational section of $\mathcal{L}$. For any $p \in C$, recall the valuation of s at p ($\nu_p(s) \in \mathbb{Z}$). (Pick any local section t of $\mathcal{L}$ not vanishing at p. Then $s/t \in FF(C)$. $\nu_p(s) := \nu_p(s/t)$. We can show that this is well-defined.)

Define $\deg(\mathcal{L}, s)$ (where s is a non-zero rational section of $\mathcal{L}$) to be the number of zeros minus the number of poles, counted with appropriate multiplicity. (In other words, each point contributes the valuation at that point times the degree of the field extension.) We'll show that this is independent of s. (Note that we need the projective hypothesis: the sections x and 1 of the structure sheaf on $\mathbb{A}^1$ have different degrees.)

Notice that $\deg(\mathcal{L}, s)$ is additive under products: $\deg(\mathcal{L}, s) + \deg(\mathcal{M}, t) = \deg(\mathcal{L} \otimes \mathcal{M}, s \otimes t)$. Thus to show that $\deg(\mathcal{L}, s) = \deg(\mathcal{L}, t)$, we need to show that $\deg(\mathcal{O}_C, s/t) = 0$. Hence it suffices to show that $\deg(\mathcal{O}_C, u) = 0$ for a non-zero rational function u on C. Then u gives a rational map $C \dashrightarrow \mathbb{P}^1$. By our recent work (Proposition 2.1 above), this can be extended to a morphism $C \to \mathbb{P}^1$. The preimage of 0 is the number of 0's, and the preimage of ∞ is the number of ∞'s. But these are the same by our previous discussion of degree of a morphism! Finally, suppose $p \mapsto 0$. I claim that the valuation of u at p times the degree of the field extension is precisely the contribution of p to $u^{-1}(0)$. (A similar computation for ∞ will complete the proof of the desired result.) This is because the contribution of p to $u^{-1}(0)$ is precisely

$$\dim_k \mathcal{O}_{C,p}/(u) = \dim_k \mathcal{O}_{C,p}/\mathfrak{m}^{\nu_p(u)} = \nu_p(u) \dim_k \mathcal{O}_{C,p}/\mathfrak{m}.$$

□

We can define the degree of an invertible sheaf $\mathcal{L}$ on an integral *singular* projective curve C as follows: if $\nu : \tilde{C} \to C$ be the normalization, let $\deg_C \mathcal{L} := \deg_{\tilde{C}} \nu^* \mathcal{L}$. Notice that if s is a meromorphic section that has neither zeros nor poles at the singular points of C, then $\deg_C \mathcal{L}$ is still the number of zeros minus the number of poles (suitably counted), because the zeros and poles of $\nu^* \mathcal{L}$ are just the same as those of $\mathcal{L}$.

3.1. *Exercise.* Suppose $f : C \to C'$ is a degree d morphism of integral projective nonsingular curves, and $\mathcal{L}$ is an invertible sheaf on C'. Show that $\deg_C f^* \mathcal{L} = d \deg_{C'} \mathcal{L}$.

3.2. Degree of a Cartier divisor on a curve.

I said the following in class 30. (I've repeated this in the class 30 notes.)

Suppose D is an effective Cartier divisor on a projective curve, or a Cartier divisor on a projective nonsingular curve (over a field k). (I should really say: suppose D is a Cartier divisor on a projective curve, but I don't think I defined Cartier divisors in that generality.) Then define the *degree* of D (denoted $\deg D$) to be the degree of the corresponding invertible sheaf.

Exercise. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

4. CECH COHOMOLOGY OF QUASICOHERENT SHEAVES

One idea behind the cohomology of quasicoherent sheaves is as follows. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, we know that

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X).$$

In other words, $\Gamma(X, \cdot)$ is a left-exact functor. We dream that this is something called H^0, and that this sequence continues off to the right, giving a long exact sequence in cohomology. (In general, whenever we see a left-exact or right-exact functor, we should hope for this, and in most good cases our dreams are fulfilled. The machinery behind this is sometimes called *derived functor cohomology*, which we may discuss in the third quarter.)

We'll show that these cohomology groups exist. Before defining them explicitly, we first describe their important properties.

Suppose X is an R-scheme. Assume throughout that X is separated and quasicompact. Then for each quasicoherent sheaf $\mathcal{F}$ on X, we'll define R-modules $H^i(X, \mathcal{F})$. (In particular, if $R = k$, they are k-vector spaces.) First, $\boxed{H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})}$. Each H^i will be a contravariant functor in the space X, and a $\boxed{\text{covariant functor in the sheaf } \mathcal{F}}$. The functor H^i behaves well under direct sums: $\boxed{H^i(X, \oplus_j \mathcal{F}_j) = \oplus_j H^i(X, \mathcal{F}_j)}$. (We will need infinite sums, not just finite sums.) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we have a $\boxed{\text{long exact sequence}}$

$$\begin{aligned} 0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \\ \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to \cdots. \end{aligned}$$

(The maps $H)(X, ?) \to H^i(X, ??)$ will be those coming from covariance; the *connecting homomorphisms* $H^i(x, \mathcal{H}) \to H^{i+1}(X, \mathcal{F})$ will have to be defined.) We'll see that if X can be covered by n affines, then $\boxed{H^i(X, \mathcal{F}) = 0}$ for $i \geq n$ for all $\mathcal{F}$, i. (In particular, all higher quasicoherent cohomology groups on affine schemes vanish.) If $X \hookrightarrow Y$ is a closed immersion, and $\mathcal{F}$ is a quasicoherent sheaf on X, then $\boxed{H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F})}$. (We'll care about this particularly in the case when $X \subset Y = \mathbb{P}^N_R$, which will let us reduce calculations on arbitrary projective R-schemes to calculations on $\mathbb{P}^n_R$.)

We will also identify the cohomology of all the invertible sheaves on $\mathbb{P}^n_R$:

4.1. *Proposition.* —

- $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free* R*-module of rank* $\binom{n+m}{n}$ *if* $i = 0$ *and* $m \geq 0$*, and* 0 *otherwise.*
- $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free* R*-module of rank* $\binom{-m-1}{-n-m-1}$ *if* $m \leq -n-1$*, and* 0 *otherwise.*
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m)) = 0$ *if* $0 < i < n$.

It is more helpful to say the following imprecise statement: $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \dots, x_n$ (with R-coefficients), and $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \dots, x_n$, where in each monomial, each x_i appears with degree at most -1.

We'll prove this next day.

Here are some features of this Proposition that I wish to point out, that will be the first appearances of things that we'll prove later.

- The cohomology of these bundles vanish above the dimension of the space *if* $R = k$; we'll generalize this for $\operatorname{Spec} R$, and even more, in before long.
- These cohomology groups are always finitely-generated R modules.
- The top cohomology group vanishes for $m > -n-1$. (This is a first appearance of "Kodaira vanishing".)
- The top cohomology group is "1-dimensional" for $m = -n-1$ if $R = k$. This is the first appearance of a dualizing sheaf.
- We have a natural duality $H^i(X, \mathcal{O}(m)) \times H^{n-i}(X, \mathcal{O}(-n-1-m)) \to H^n(X, \mathcal{O}(-n-1))$. This is the first appearance of Serre duality.

I'd like to use all these properties to prove things, so you'll see how handy they are. We'll worry later about defining cohomology, and proving these properties.

When we discussed global sections, we worked hard to show that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_R$ we could find a surjection $\mathcal{O}(m)^{\oplus j} \to \mathcal{F}$, which yields the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}(m)^{\oplus j} \to \mathcal{F} \to 0 \tag{1}$$

for some coherent sheaf $\mathcal{G}$. We can use this to prove the following.

4.2. *Theorem. — (i) For any coherent sheaf $\mathcal{F}$ on a projective R-scheme where R is Noetherian, $h^i(X, \mathcal{F})$ is a finitely generated R-module. (ii) (Serre vanishing) Furthermore, for $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ for all i, even without Noetherian hypotheses.*

Proof. Because cohomology of a closed scheme can be computed on the ambient space, we may reduce to the case $X = \mathbb{P}^n_R$.

(i) Consider the long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{P}^n_R, \mathcal{G}) \arrow[r] & H^0(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \arrow[r] & H^0(\mathbb{P}^n_R, \mathcal{F}) \arrow[r] & {} \\
& H^1(\mathbb{P}^n_R, \mathcal{G}) \arrow[r] & H^1(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \arrow[r] & H^1(\mathbb{P}^n_R, \mathcal{F}) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{n-1}(\mathbb{P}^n_R, \mathcal{G}) \arrow[r] & H^{n-1}(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \arrow[r] & H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \arrow[r] & {} \\
& H^n(\mathbb{P}^n_R, \mathcal{G}) \arrow[r] & H^n(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \arrow[r] & H^n(\mathbb{P}^n_R, \mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

The exact sequence ends here because $\mathbb{P}^n_R$ is covered by $n+1$ affines. Then $H^n(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j})$ is finitely generated by Proposition 4.1, hence $H^n(\mathbb{P}^n_R, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. Hence in particular, $H^n(\mathbb{P}^n_R, \mathcal{G})$ is finitely generated. As $H^{n-1}(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j})$ is finitely generated, and $H^n(\mathbb{P}^n_R, \mathcal{G})$ is too, we have that $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. We continue inductively downwards.

(ii) Twist (4.1) by $\mathcal{O}(N)$ for $N \gg 0$. Then $H^n(\mathbb{P}^n_R, \mathcal{O}(m+N)^{\oplus j}) = 0$, so $H^n(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$. Translation: for any coherent sheaf, its top cohomology vanishes once you twist by $\mathcal{O}(N)$ for N sufficiently large. Hence this is true for $\mathcal{G}$ as well. Hence from the long exact sequence, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$ for $N \gg 0$. As in (i), we induct downwards, until we get that $H^1(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$. (The induction proceeds no further, as it is *not* true that $H^0(\mathbb{P}^n_R, \mathcal{O}(m+N)^{\oplus j}) = 0$ for large N — quite the opposite. $\square$

Exercise for those who like working with non-Noetherian rings: Prove part (i) in the above result without the Noetherian hypotheses, assuming only that R is a coherent R-module (it is "coherent over itself"). (Hint: induct downwards as before. The order is as follows: $H^n(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_R, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, etc.)

In particular, we have proved the following, that we would have cared about even before we knew about cohomology.

4.3. *Corollary. — Any projective k-scheme has a finite-dimensional space of global sections. More generally, if $\mathcal{F}$ is a coherent sheaf on a projective R-scheme, then $h^0(X, \mathcal{F})$ is a finitely generated R-module.*

This is true more generally for proper k-schemes, not just projective k-schemes, but I won't give the argument here.

Here is another a priori interesting consequence:

4.4. *Corollary. — If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on projective X with $\mathcal{F}$ coherent, then for $n \gg 0$, $0 \to H^0(X, \mathcal{F}(n)) \to H^0(X, \mathcal{G}(n)) \to H^0(X, \mathcal{H}(n)) \to 0$ is also exact.*

(Proof: for $n \gg 0$, $H^1(X, \mathcal{F}(n)) = 0$.)

This result can also be shown directly, without the use of cohomology.

5. Proving the things you need to know

As you read this, you should go back and check off all the facts, to make sure that I've shown all that I've promised.

5.1. Cech cohomology. Works nicely here. In general: take finer and finer covers. Here we take a single cover.

Suppose X is quasicompact and separated, e.g. X is quasiprojective over A. In particular, X may be covered by a finite number of affine open sets, and the intersection of any two affine open sets is also an affine open set; these are the properties we will use. Suppose $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{U} = \{U_i\}_{i=1}^n$ is a *finite* set of affine open sets of X whose union is U. For $I \subset \{1, \dots, n\}$ define $U_i = \cap_{i \in I} U_i$. It is affine by the separated hypothesis. **Define** $H^i_{\mathcal{U}}(U, \mathcal{F})$ to be the ith cohomology group of the complex

$$0 \to \bigoplus_{\substack{|I| = 1 \\ I \subset \{1, \dots, n\}}} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{\substack{|I| = i \\ I \subset \{1, \dots, n\}}} \mathcal{F}(U_I) \to \bigoplus_{\substack{|I| = i+1 \\ I \subset \{1, \dots, n\}}} \mathcal{F}(U_I) \to \cdots . \tag{2}$$

Note that if X is an R-scheme, then $H^i_{\mathcal{U}}(X, \mathcal{F})$ is an R-module. Also $H^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

5.2. *Exercise.* Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. Show that we get a long exact sequence of cohomology. (Note that this applies in our case!)

I ended by stating the following result, which we will prove next day.

5.3. *Theorem/Definition. — Recall that X is quasicompact and separated. $H^i_{\mathcal{U}}(U, \mathcal{F})$ is independent of the choice of (finite) cover $\{U_i\}$. More precisely,*

- (*) *for all k, for any two covers $\{U_i\} \subset \{V_i\}$ of size at most k, the maps $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ induced by the natural maps of complex* (2) *are isomorphisms.*

Define the Cech cohomology group $H^i(X, \mathcal{F})$ to be this group.

I needn't have stated in terms of some k; I've stated it in this way so I can prove it by induction.

(For experts: we'll get natural quasiisomorphisms of Cech complexes for various $\mathcal{U}$.)

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