FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 27

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CONTENTS

Last day: proper morphisms.

Today: a little more propriety. Rational maps. Curves.

(These notes include some facts discussed in class 28, for the sake of continuity.)

1. PROPER MORPHISMS

Last day we mostly proved:

1.1. *Theorem. — Projective morphisms are proper.*

We had reduced it to the following fact:

1.2. *Proposition. —* $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ *is a closed morphism.*

Proof. Suppose $Z \hookrightarrow \mathbb{P}^n_A$ is a closed *subset*. We wish to show that $\pi(Z)$ is closed.

Suppose $y \notin \pi(Z)$ is a *closed* point of $\operatorname{Spec} A$. We'll check that there is a distinguished open neighborhood $D(f)$ of y in $\operatorname{Spec} A$ such that $D(f)$ doesn't meet $\pi(Z)$. (If we could show this for *all* points of $\pi(Z)$, we would be done. But I prefer to concentrate on closed points for now.) Suppose y corresponds to the maximal ideal $\mathfrak{m}$ of A. We seek $f \in A - \mathfrak{m}$ such that $\pi^* f$ vanishes on Z.

A picture helps here, but I haven't put it in the notes.

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Let $U_0, \ldots, U_n$ be the usual affine open cover of $\mathbb{P}^n_A$. The closed subsets $\pi^{-1}y$ and Z do not intersect. On the affine open set U_i, we have two closed subsets that do not intersect, which means that the ideals corresponding to the two open sets generate the unit ideal, so in the ring of functions on U_i, we can write

$$1 = a_i + \sum m_{ij} g_{ij}$$

where $m_{ij} \in \mathfrak{m}$, and a_i vanishes on Z. Note that $a_i, g_{ij} \in A[x_{0/i}, x_{1/i}, \ldots, x_{n/i}]$. So by multiplying by a sufficiently high power x_i^n of x_i, we have an equality

$$x_i^N = a_i' + \sum m_{ij} g_{ij}'$$

on U_i, where both sides are expressions in $A[x_0, \ldots, x_n]$. We may take N large enough so that it works for all i. Thus for N' sufficiently large, we can write any monomial in $x_1, \ldots, x_n$ of degree N' as something vanishing on Z plus a linear combination of elements of $\mathfrak{m}$ times other polynomials. Hence if $S_* = A[x_0, \ldots, x_n]$,

$$S_{N'} = I(Z)_{N'} + \mathfrak{m} S_{N'}$$

where $I(Z)_*$ is the graded ideal of functions vanishing on Z. Hence by Nakayama's lemma, there exists $f \in A - \mathfrak{m}$ such that

$$f S_{N'} \subset I(Z)_{N'}.$$

Thus we have found our desired f!

We are now ready to tackle the proposition in general. Suppose $y \in \operatorname{Spec} A$ is no longer necessarily a closed point, and say $y = [\mathfrak{p}]$. Then we apply the same argument in $\operatorname{Spec} A_{\mathfrak{p}}$. We get $S_{N'} \otimes A_{\mathfrak{p}} = I(Z)_{N'} \otimes A_{\mathfrak{p}} + \mathfrak{m} S_{N'} \otimes A_{\mathfrak{p}}$, from which $g(S_{N'}/I(Z)_{N'}) \otimes A_{\mathfrak{p}} = 0$ for some $g \in A_{\mathfrak{p}} - \mathfrak{p} A_{\mathfrak{p}}$, from which $(S_{N'}/I(Z)_{N'}) \otimes A_{\mathfrak{p}} = 0$. Now $S_{N'}$ is a finitely generated A-module, so there is some $f \in R - \mathfrak{p}$ with $f S_N \subset I(Z)$ (if the module-generators of $S_{N'}$, and $f_1, \ldots, f_a$ are annihilate the generators respectively, then take $f = \prod f_i$), so once again we have found $D(f)$ containing $\mathfrak{p}$, with (the pullback of) f vanishing on Z. $\square$

2. Scheme-theoretic closure, and scheme-theoretic image

Have I defined scheme-theoretic closure of a locally closed subscheme $W \hookrightarrow Y$? I think I have neglected to. It is the smallest closed subscheme of Y containing W. *Exercise.* Show that this notion is well-defined. More generally, if $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. *Exercise.* Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\coprod_{n \geq 0} \operatorname{Spec} k[t]/t^n \to \operatorname{Spec} k[t]$. (I suspect that such a pathology cannot occur for finite type morphisms of Noetherian schemes, but I haven't investigated.)

3. Rational maps

This is a very old topic, near the beginning of any discussion of varieties. It has appeared late for us because we have just learned about separatedness.

For this section, I will suppose that X and Y are integral and separated, although these notions are often useful in more general circumstances. The interested reader should consider the first the case where the schemes in question are reduced and separated (but not necessarily irreducible). Many notions can make sense in more generality (without reducedness hypotheses for example), but I'm not sure if there is a widely accepted definition.

A key example will be irreducible varieties, and the language of rational maps is most often used in this case.

A **rational map** $X \dashrightarrow Y$ is a morphism on a dense open set, with the equivalence relation: $(f : U \to Y) \sim (g : V \to Y)$ if there is a dense open set $Z \subset U \cap V$ such that $f|_Z = g|_Z$. (We will soon see that we can add: if $f|_{U\cap V} = g|_{U\cap V}$.)

An obvious example of a rational map is a morphism. Another example is a rational function, which is a rational map to $\mathbb{A}^1_{\mathbb{Z}}$ (*easy exercise*).

3.1. *Exercise.* Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant.

3.2. *Easy exercise.* Show that dominant rational maps give morphisms of function fields in the opposite direction. (This was problem 37 on problem set 9.)

It is not true that morphisms of function fields give dominant rational maps, or even rational maps. For example, $k[x]$ and $\operatorname{Spec} k(x)$ have the same function field ($k(x)$), but there is no rational map $\operatorname{Spec} k[x] \dashrightarrow k(x)$. Reason: that would correspond to a morphism from an open subset U of $\operatorname{Spec} k[x]$, say $k[x, 1/f(x)]$, to $k(x)$. But there is no map of rings $k(x) \to k[x, 1/f(x)]$ for any one $f(x)$.

However, this is true in the case of varieties (see Proposition 3.4 below).

A rational map $f : X \to Y$ is said to be *birational* if it is dominant, and there is another morphism (a "rational inverse") that is also dominant, such that $f \circ g$ is (in the same equivalence class as) the identity on Y, and $g \circ f$ is (in the same equivalence class as) the identity on X.

A morphism is **birational** if it is birational as a rational map. We say X and Y are *birational* to each other if there exists a birational map $X \dashrightarrow Y$. This is the same as our definition before. Birational maps induce isomorphisms of function fields.

3.3. *Important Theorem. — Two S-morphisms $f_1, f_2 : U \to Z$ from a reduced scheme to a separated S-scheme agreeing on a dense open subset of U are the same.*

Note that this generalizes the easy direction of the valuative criterion of separatedness (which is the special case where U is Spec of a discrete valuation ring — which consists of two points — and the dense open set is the generic point).

It is useful to see how this breaks down when we give up reducedness of the base or separatedness of the target. For the first, consider the two maps $\operatorname{Spec} k[x,y]/(x^2, xy) \to \operatorname{Spec} k[t]$, where we take f_1 given by $t \mapsto y$ and f_2 given by $t \mapsto y + x$; f_1 and f_2 agree on the distinguished open set $D(y)$. (A picture helps here!) For the second, consider the two maps from $\operatorname{Spec} k[t]$ to the line with the doubled origin, one of which maps to the "upper half", and one of which maps to the "lower half". these to morphisms agree on the dense open set $D(f)$.

Proof.

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "\text{cl. imm.}"'] & Y \arrow[d, "\Delta"] \\
U \arrow[r, "{(f_1,f_2)}"] & Y \times Y
\end{tikzcd}$$

We have a closed subscheme of U containing the generic point. It must be all of U. □

Consequence 1. Hence (as X is reduced and Y is separated) if we have two morphisms from open subsets of X to Y, say $f : U \to Y$ and $g : V \to Y$, and they agree on a dense open subset $Z \subset U \cap V$, then they necessarily agree on $U \cap V$.

Consequence 2. Also: a rational map has a largest *domain of definition* on which $f : U \dashrightarrow Y$ is a morphism, which is the union of all the domains of definition.

In particular, a rational function from a reduced scheme has a largest *domain of definition*.

We define the *graph* of a rational map $f : X \dashrightarrow Y$ as follows: let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion. *Exercise.* Show that this is independent of the choice of U.

Here is a handy diagram involving the graph of a rational map:

$$\begin{tikzcd}
\Gamma \arrow[r, hook] & X \times Y \arrow[dl] \arrow[dr] & \\
X \arrow[u, dashed] & & Y
\end{tikzcd}$$

(that "up arrow" should be dashed).

We now prove a Proposition promised earlier.

3.4. *Proposition. — Suppose* X, Y *are irreducible varieties, and we are given* $f^{\#} : FF(Y) \hookrightarrow FF(Y)$. *Then there exists a dominant rational map* $f : X \dashrightarrow Y$ *inducing* $f^{\#}$.

Proof. By replacing Y with an affine open set, we may assume Y is affine, say $Y = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then we have $x_1, \dots, x_n \in K(X)$. Let U be an open subset of the domains of definition of these rational functions. Then we get a morphism $U \to \mathbb{A}^n_k$. But this morphism factors through $Y \subset \mathbb{A}^n$, as $x_1, \dots, x_n$ satisfy all the relations $f_1, \dots, f_r$. $\square$

3.5. *Exercise.* Let K be a finitely generated field extension of transcendence degree m over k. Show there exists an irreducible k-variety W with function field K. (Hint: let $x_1, \dots, x_n$ be generators for K over k. Consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \dots, t_n]$ given by the ring map $t_i \mapsto x_i$. Take the scheme-theoretic closure of the image.)

3.6. *Proposition. — Suppose X and Y are integral and separated (our standard hypotheses from last day). Then X and Y are birational if and only if there is a dense=non-empty open subscheme U of X and a dense=non-empty open subscheme V of Y such that $U \cong Y$.*

This gives you a good idea of how to think of birational maps.

3.7. *Exercise.* Prove this. (Feel free to consult Iitaka or Hartshorne (Corollary I.4.5).)

4. Examples of rational maps

We now give a bunch of examples Here are some examples of rational maps, and birational maps. A recurring theme is that domains of definition of rational maps to projective schemes extend over nonsingular codimension one points. We'll make this precise when we discuss curves shortly.

(A picture is helpful here.) The first example is how you find a formula for Pythagorean triples. Suppose you are looking for rational points on the circle C given by $x^2 + y^2 = 1$. One rational point is $p = (1, 0)$. If q is another rational point, then pq is a line of rational (non-infinite) slope. This gives a rational map from the conic to $\mathbb{A}^1$. Conversely, given a line of slope m through p, where m is rational, we can recover q as follows: $y = m(x-1)$, $x^2 + y^2 = 1$. We substitute the first equation into the second, to get a quadratic equation in x. We know that we will have a solution $x = 1$ (because the line meets the circle at $(x, y) = (1, 0)$), so we expect to be able to factor this out, and find the other factor. This indeed works:

$$\begin{aligned} x^2 + (m(x-1))^2 &= 1 \\ (m^2+1)x^2 + (-2)x + (m^2-1) &= 0 \\ (x-1)((m^2+1)x - (m^2-1)) &= 0 \end{aligned}$$

The other solution is $x = (m^2-1)/(m^2+1)$, which gives $y = 2m/(m^2+1)$. Thus we get a birational map between the conic C and $\mathbb{A}^1$ with coordinate m, given by $f : (x, y) \mapsto y/(x-1)$ (which is defined for $x \neq 1$), and with inverse rational map given by $m \mapsto ((m^2-1)/(m^2+1), 2m/(m^2+1))$ (which is defined away from $m^2 + 1 = 0$).

We can extend this to a rational map $C \dashrightarrow \mathbb{P}^1$ via the inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. Then f is given by $(x, y) \mapsto [y; x-1]$. (Remember that we give maps to projective space by giving sections of line bundles — in this case, we are using the structure sheaf.) We then have an interesting question: what is the domain of definition of f? It appears to be defined everywhere except for where $y = x - 1 = 0$, i.e. everywhere but p. But in fact it can be extended over p! Note that $(x, y) \mapsto [x+1; -y]$ (where $(x, y) \neq (-1, y)$) agrees with f on their common domains of definition, as $[x+1; -y] = [y; x-1]$. Hence this rational map can be extended farther than we at first thought. This will be a special case of a result we'll see later today.

(For the curious: we are working with schemes over $\mathbb{Q}$. But this works for any scheme over a field of characteristic not 2. What goes wrong in characteristic 2?)

4.1. *Exercise.* Use the above to find a "formula" for all Pythagorean triples.

4.2. *Exercise.* Show that the conic $x^2 + y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (Presumably this is true for any ring in which 2 is invertible too...)

In fact, any conic in $\mathbb{P}^2_k$ with a k-valued point (i.e. a point with residue field k) is isomorphic to $\mathbb{P}^1_k$. (This hypothesis is certainly necessary, as $\mathbb{P}^1_k$ certainly has k-valued points. $x^2 + y^2 + z^2 = 0$ over $k = \mathbb{R}$ gives an example of a conic that is not isomorphic to $\mathbb{P}^1_k$.)

4.3. *Exercise.* Find all rational solutions to $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

You will obtain a rational map to $\mathbb{P}^1$ that is not defined over the node $x = y = 0$, and *can't* be extended over this codimension 1 set. This is an example of the limits of our future result showing how to extend rational maps to projective space over codimension 1 sets: the codimension 1 sets have to be nonsingular. More on this soon!

4.4. *Exercise.* Use something similar to find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

4.5. *Exercise (a first view of a blow-up).* Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Show that $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ is nonsingular. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism

$\mathrm{Bl}_{(0,0)} \mathbb{A}_k^2 \to \mathbb{A}_k^2$. Show that the fiber over $(0,0)$ is an effective Cartier divisor. It is called the *exceptional divisor*.

4.6. *Exercise (the Cremona transformation, a useful classical construction).* Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x;y;z] \to [1/x;1/y;1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

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