FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 25

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CONTENTS

Last day: Normalization (in a finite field extension), "sheaf Spec", "sheaf Proj", projective morphisms.

Today: separatedness, definition of variety.

0.1. Here is a notion I should have introduced earlier: *induced reduced subscheme structure*. Suppose X is a scheme, and S is a *closed subset* of X. Then there is a unique reduced closed subscheme Z of X "supported on S". More precisely, it can be defined by the following universal property: for any morphism from a *reduced* scheme Y to X, whose image lies in S (as a set), this morphism factors through Z uniquely. Over an affine $X = \mathrm{Spec}\, R$, we get $\mathrm{Spec}\, R/I(S)$. (Exercise: verify this.) For example, if S is the entire underlying set of X, we get X^{red}.

1. SEPARATED MORPHISMS

We will now describe a very useful notion, that of morphisms being *separated*. Separatedness is one of the definitions in algebraic geometry (like flatness) that seems initially unmotivated, but later turns out to be the answer to a large number of desiderata.

Here are some initial reasons. First, in some sense it is the analogue of Hausdorff. A better description is the following: if you take the definition I'm about to give you and apply it to the "usual" topology, you'll get a correct (if unusual) definition of Hausdorffness. The reason this doesn't give Hausdorffness in the category of schemes is because the topology on the product is not the product topology. (An earlier exercise was to show that $\mathbb{A}^2_k$ does not have the product topology on $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$.) One benefit of this definition is that we will be finally ready to define a *variety*, in a way that corresponds to the classical definition.

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Second, a separated morphism has the property that the intersection of a two affine open sets is affine, which is precisely the odd hypothesis needed to make Cech cohomology work.

A third motivation is that nasty line with doubled origin, which is a counterexample to many statements one might hope are true. The line with double origin is not separated, and by adding a separatedness hypothesis, the desired statements turn out to be true.

A fourth motivation is to give a good foundation for the notion of rational maps, which we will discuss shortly.

A lesson arising from the construction is the importance of the diagonal morphism. More precisely given a morphism $X \to Y$, nice consequences can be leveraged from good behavior of the diagonal morphism $\delta : X \to X \times_Y X$, usually through fun diagram chases. This is a lesson that applies across many fields of mathematics. (Another nice gift the diagonal morphism: it will soon give us a good algebraic definition of differentials.)

1.1. *Proposition. — Let* $X \to Y$ *be a morphism of schemes. Then the diagonal morphism* $\delta : X \to X \times_Y X$ *is a locally closed immersion.*

This locally closed subscheme of $X \times_Y X$ (the diagonal) will be denoted Δ.

Proof. We will describe a union of open subsets of $X \times_Y X$ covering the image of X, such that the image of X is a closed immersion in this union.

1.2. Say Y is covered with affine opens V_i and X is covered with affine opens U_{ij}, with $\pi : U_{ij} \to V_i$. Then the diagonal is covered by $U_{ij} \times_{V_i} U_{ij}$. (Any point $p \in X$ lies in some U_{ij}; then $\delta(p) \in U_{ij} \times_{V_i} U_{ij}$.) Note that $\delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij}$: $U_{ij} \times_{V_i} U_{ij} \cong U_{ij} \times_Y U_{ij}$ because $V_i \hookrightarrow Y$ is a monomorphism. Then because open immersions behave well with respect to base change, we have the fiber diagram

$$\begin{array}{ccc} U_{ij} & \longrightarrow & X \\ \downarrow & & \downarrow \\ U_{ij} \times_Y X & \longrightarrow & X \times_Y X \end{array}$$

from which $\delta^{-1}(U_{ij} \times_Y X) = U_{ij}$. As $\delta^{-1}(U_{ij} \times_Y U_{ij})$ contains U_{ij}, we must have $\delta^{-1}(U_{ij} \times_Y U_{ij}) = U_{ij}$.

Finally, we'll check that $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$ is a closed immersion. Say $V_i = \operatorname{Spec} S$ and $U_{ij} = \operatorname{Spec} R$. Then this corresponds to the natural ring map $R \times_S R \to R$, which is obviously surjective. □

(A picture is helpful here.)

Note that the open subsets we described may not cover $X \times_Y X$, so we have not shown that δ is a closed immersion.

1.3. *Definition.* A morphism $X \to Y$ is said to be **separated** if the diagonal morphism $\delta : X \to X \times_Y X$ is a closed immersion. If R is a ring, an R-scheme X is said to be *separated over* R if the structure morphism $X \to \operatorname{Spec} R$ is separated. When people say that a scheme (rather than a morphism) X is separated, they mean implicitly that some morphism is separated. For example, if they are talking about R-schemes, they mean that X is separated over R.

Thanks to Proposition 1.1, a morphism is separated if and only if the image of the diagonal morphism is closed.

1.4. *Important easy exercise.* Show that open immersions and closed immersions are separated. (Answer: Show that monomorphisms are separated. Open and closed immersions are monomorphisms, by earlier exercises. Alternatively, show this by hand.)

1.5. *Important easy exercise.* Show that every morphism of affine schemes is separated. (Hint: this was essentially done in Proposition 1.1.)

I'll now give you an example of something separated that is not affine. The following single calculation will eventually easily imply that all quasiprojective morphisms are separated.

1.6. *Proposition.* — $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ *is separated.*

(The identical argument holds with $\mathbb{Z}$ replaced by any ring.)

Proof. We cover $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ with open sets of the form $U_i \times U_j$, where $U_0, \ldots, U_n$ form the "usual" affine open cover. The case $i = j$ was taken care of before, in the proof of Proposition 1.1. For $i \neq j$, we may take $i = 0, j = n$. Then

$$U_0 \times_{\mathbb{Z}} U_n \cong \operatorname{Spec} \mathbb{Z}[x_{1/0}, \ldots, x_{n/0}, y_{0/n}, \ldots, y_{n-1/n}],$$

and the image of the diagonal morphism meets this open set in the closed subscheme $y_{0/n} x_{n/0} = 1$, $x_{i/0} = x_{n/0} y_{i/n}$, $y_{j/n} = y_{0/n} x_{j/0}$. □

1.7. *Exercise.* Verify the last sentence of the proof. Note that you should check that the diagonal morphism restricted to this open set has source $U_0 \cap U_n$; see §1.2.

1.8. *Exercise.* Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed. (Another argument is given below, in Exercise 1.28.)

We finally define then notion of variety!

1.9. *Definition.* A **variety** over a field k is defined to be a reduced, separated scheme of finite type over k. We may use the language k-variety.

Example: a reduced finite type affine k-scheme is a variety. In other words, to check if $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ is a variety, you need only check reducedness.

Notational caution: In some sources (including, I think, Mumford), the additional condition of irreducibility is imposed. We will not do this. Also, it is often assumed that k is algebraically closed. We will not do this either.

Here is a very handy consequence of separatedness!

1.10. *Proposition. — Suppose $X \to \operatorname{Spec} R$ is a separated morphism to an affine scheme, and U and V are affine open sets of X. Then $U \cap V$ is an affine open subset of X.*

We'll prove this shortly.

Consequence: if $X = \operatorname{Spec} A$, then the intersection of any two affine opens is open (just take $R = \mathbb{Z}$ in the above proposition). This is certainly not an obvious fact! We know that the intersection of any two distinguished affine open sets is affine (from $D(f) \cap D(g) = D(fg)$), but we have very little handle on affine open sets in general.

Warning: this property does not characterize separatedness. For example, if $R = \operatorname{Spec} k$ and X is the line with doubled origin over k, then X also has this property. This will be generalized slightly in Exercise 1.31.

Proof. Note that $(U \times_{\operatorname{Spec} R} V) \cap \Delta = U \cap V$, where Δ is the diagonal. (This is clearest with a figure. See also §1.2.)

$U \times_{\operatorname{Spec} R} V$ is affine ($\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} T = \operatorname{Spec} S \otimes_R T$), and Δ is a closed subscheme of an affine scheme, and hence affine. □

1.11. Sample application: The graph morphism.

1.12. *Definition.* Suppose $f : X \to Y$ is a morphism of Z-schemes. The morphism $\Gamma : X \to X \times_Z Y$ given by $\Gamma = (\mathrm{id}, f)$ is called the **graph morphism**.

1.13. *Proposition. — Show that Γ is a locally closed immersion. Show that if Y is a separated Z-scheme (i.e. the structure morphism $Y \to Z$ is separated), then Γ is a closed immersion.*

This will be generalized in Exercise 1.29.

Proof by diagram.

$$\begin{array}{ccc} X & \longrightarrow & X \times_Z Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta} & Y \times_Z Y \end{array}$$

□

1.14. Quasiseparated morphisms.

We now define a handy relative of separatedness, that is also given in terms of a property of the diagonal morphism, and has similar properties. The reason it is less famous is because it automatically holds for the sorts of schemes that people usually deal with. We say a morphism $f : X \to Y$ is **quasiseparated** if the diagonal morphism $\delta : X \to X \times_Y X$ is quasicompact. I'll give a more insightful translation shortly, in Exercise 1.15.

Most algebraic geometers will only see quasiseparated morphisms, so this may be considered a very weak assumption. Here are two large classes of morphisms that are quasiseparated. (a) As closed immersions are quasicompact (not hard), separated implies quasiseparated. (b) If X is a Noetherian scheme, then any morphism to another scheme is quasicompact (not hard; *Exercise*), so any $X \to Y$ is quasiseparated. Hence those working in the category of Noetherian schemes need never worry about this issue.

It is the following characterization which makes quasiseparatedness a useful hypothesis in proving theorems.

1.15. *Exercise.* Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} R$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} R$, $U \cap V$ is a *finite* union of affine open sets.

1.16. *Exercise.* Here is an example of a nonquasiseparated scheme. Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated.

In particular, the condition of quasiseparatedness is often paired with quasicompactness in hypotheses of theorems. A morphism $f : X \to Y$ is quasicompact and quasiseparated if and only if the preimage of any affine open subset of Y is a *finite* union of affine open sets in X, whose pairwise intersections are all *also* finite unions of affine open sets.

This strong finiteness assumption can be very useful, as the following result shows:

1.17. *Proposition. — If $X \to Y$ is a quasicompact, quasiseparated morphism, and $\mathcal{F}$ is quasicoherent sheaf on X, show that $f_*\mathcal{F}$ is a quasicoherent sheaf on Y.*

Proof. The proof we gave earlier (Theorem 2.2 of Class 20) applies without change. We just didn't have the name "quasiseparated" to attach to these hypothesis. □

1.18. *Theorem. — Both separatedness and quasiseparatedness are preserved by base change.*

Proof. Suppose

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a fiber square. We will show that if $Y \to Z$ is separated or quasiseparated, then so is $W \to X$. The reader should verify (using only category theory!) that

$$\begin{array}{ccc} W & \xrightarrow{\delta_W} & W \times_X W \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta_Y} & Y \times_Z Y \end{array}$$

is a fiber diagram. As the property of being a closed immersion is preserved by base change (shown earlier when we showed many properties are well behaved under base change) , if δ_Y is a closed immersion, so is δ_X.

Quasiseparatedness follows in the identical manner, as quasicompactness is also preserved by base change. □

1.19. *Proposition. — The condition of being separated is local on the target. Precisely, a morphism* $f : X \to Y$ *is separated if and only if for any cover of* Y *by open subsets* U_i, $f^{-1}(U_i) \to U_i$ *is separated for each* i.

Hence affine morphisms are separated, by Exercise 1.5. (Thus finite morphisms are separated.)

Proof. If $X \to Y$ is separated, then for any $U_i \hookrightarrow Y$, $f^{-1}(U_i) \to U_i$ is separated by Theorem 1.18. Conversely, to check if $\Delta \hookrightarrow X \times_Y X$ is a closed subset, it suffices to check this on an open cover. If $g : X \times_Y X \to Y$ is the natural morphism, our open cover U_i of Y induces an open cover $g^{-1}(U_i)$ of $X \times_Y X$. □

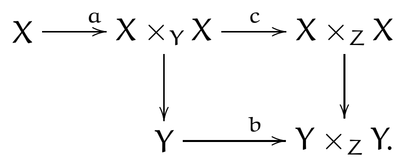
1.20. *Exercise.* Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target by an earlier exercise; use a similar argument.)

1.21. *Proposition. — The condition of being separated is closed under composition. In other words, if* $f : X \to Y$ *is separated and* $g : Y \to Z$ *is separated, then* $g \circ f : X \to Z$ *is separated.*

Proof. This is a good excuse to show you a very useful fiber diagram:

$$\boxed{\begin{array}{ccc} U \times_X V & \longrightarrow & U \times_S V \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_S X \end{array}}$$

We are given that $a : X \hookrightarrow X \times_Y X$ and $b : Y \to Y \times_Z Y$ are closed immersions, and we wish to show that $X \to X \times_Z X$ is a closed immersion. Consider the diagram



The square on the right is a fiber diagram (see the very useful diagram above). As b is a closed immersion, c is too (closed immersions behave well under fiber diagrams). Thus $c \circ a$ is a closed immersion (the composition of two closed immersions is also a closed immersion). ☐

The identical argument (with "closed immersion" replaced by "quasicompact") shows that the condition of being quasiseparated is closed under composition.

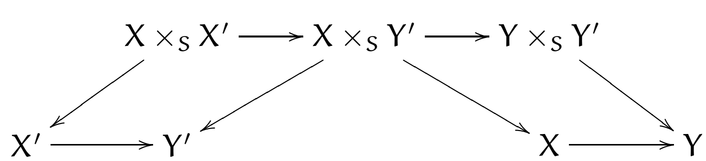
1.22. *Proposition. — Any quasiprojective morphism is separated.*

As a corollary, any reduced quasiprojective k-scheme is a k-variety.

Proof. Open immersions are separated by Exercise 1.4. Hence by Proposition 1.21, it suffices to check that projective morphisms are separated. We can check that this locally on the target by Proposition 1.19, so it suffices to check that $f : X \to Z$ where f factors through $\mathbb{P}^n_Z$, and $X \hookrightarrow \mathbb{P}^n_Z$ is a closed immersion. But closed immersions are separated, so $X \hookrightarrow \mathbb{P}^n_Z$ is separated, so it suffices to check $\mathbb{P}^n_Z \to Z$ is separated. But this is obtained by base change from $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, so we are done (as this latter morphism is separated by the previous proposition, and separatedness is preserved by base change by Proposition 1.18). ☐

1.23. *Proposition. — Suppose $f : X \to Y$ and $f' : X' \to Y'$ are separated morphisms of S-schemes. Then the product morphism $f \times f' : X \times_S X' \to Y \times_S Y'$ is separated.*

Proof. Consider the following diagram, and use the fact that separatedness is preserved under base change and composition.



☐

1.24. A very fun result.

We now come to a very useful, but bizarre-looking, result.

1.25. *Proposition. — Let* P *be a class of morphisms that is preserved by base change and composition. Suppose*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_{h}\searrow \quad \swarrow_{g} & \\ & Z & \end{array}$$

is a commuting diagram of schemes.

(a) *Suppose that the diagonal morphism* $\delta_g : Y \to Y \times_Z Y$ *is in* P *and* $h : X \to Z$ *is in* P. *The* $f : X \to Y$ *is in* P.
(b) *In particular, if closed immersions are in* P, *then if* h *is in* P *and* g *is separated, then* f *is in* P.

I like this because when you plug in different P, you get very different-looking (and non-obvious) consequences.

Here are some examples.

Locally closed immersions are separated, so part (a) applies, and the first clause always applies. In other words, if you factor a locally closed immersion $X \to Z$ into $X \to Y \to Z$, then $X \to Y$ must be a locally closed immersion.

A morphism (over $\operatorname{Spec} k$) from a projective k-scheme to a separated k-scheme is always projective.

Possibilities for P in case (b) include: finite morphisms, morphisms of finite type, projective morphisms (needed exercise: closed immersions are projective), closed immersions, affine morphisms.

Proof. By the fibered square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & X \times_Z Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta_Y} & Y \times_Z Y \end{array}$$

we see that the graph morphism $\Gamma : X \to X \times_Z Y$ is in P (Definition 1.12), as P is closed under base change. By the fibered square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{h'} & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{h} & Z \end{array}$$

the projection $h' : X \times_Z Y \to Y$ is in P as well. Thus $f = h' \circ \Gamma$ is in P □

1.26. *Exercise.* Show that a k-scheme is separated (over k) iff it is separated over $\mathbb{Z}$.

Here now are some fun and useful exercises.

1.27. *Useful exercise: The locus where two morphisms agree.* We can now make sense of the following statement. Suppose

$$f, g: \quad \begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

are two morphisms over Z. Then the locus on X where f and g agree is a locally closed subscheme of X. If $Y \to Z$ is separated, then the locus is a closed subscheme of X. More precisely, define V to be the following fibered product:

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \delta \\ X & \xrightarrow{(f,g)} & Y \times_Z Y. \end{array}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V. (Put differently: we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.) Now we come to the exercise: prove this (the sentence before the parentheses). (Hint: we get a map $g \circ h = f \circ h : W \to Y$. Use the definition of fibered product to get $W \to V$.)

1.28. *Exercise.* Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme (cf. Proposition 1.1). (Another argument was given above, in Exercise 1.8.)

1.29. *Exercise.* Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 1.13.)

1.30. *Less important exercise.* Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $X \to Y$ is in P then $X^{\text{red}} \to Y^{\text{red}}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) (Hint:

$$\begin{array}{ccccc} X^{\text{red}} & \longrightarrow & X \times_Y Y^{\text{red}} & \longrightarrow & Y^{\text{red}} \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

)

1.31. *Exercise.* Suppose $\pi : X \to Y$ is a morphism or a ring R, Y is a separated R-scheme, U is an affine open subset of X, and V is an affine open subset of Y. Show that $U \cap \pi^{-1} V$ is an affine open subset of X. (Hint: this generalizes Proposition 1.9 of the Class 25 notes. Use Proposition 1.12 or 1.13.) This will be used in the proof of the Leray spectral sequence.

2. Valuative criteria for separatedness

Describe fact that some people love. It can be useful. I've never used it. But it gives good intuition.

It is possible to verify separatedness by checking only maps from valuations rings.

We begin with a valuative criterion that applies in a case that will suffice for the interests of most people, that of finite type morphisms of Noetherian schemes. We'll then give a more general version for more general readers.

2.1. *Theorem (Valuative criterion for separatedness for morphisms of finite type of Noetherian schemes). — Suppose* $f : X \to Y$ *is a morphism of finite type of Noetherian schemes. Then* f *is separated if and only if the following condition holds. For any discrete valuation ring* R *with function field* K*, and any diagram of the form*

(1)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

(where the vertical morphism on the left corresponds to the inclusion $R \hookrightarrow K$*), there is at most one morphism* $\operatorname{Spec} R \to X$ *such that the diagram*

(2)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

commutes.

A useful thing to take away from this statement is the intuition behind it. We think of $\operatorname{Spec} R$ as a "germ of a curve", and $\operatorname{Spec} K$ as the "germ minus the origin". Then this says that if we have a map from a germ of a curve to Y, and have a lift of the map away from the origin to X, then there is at most one way to lift the map from the entire germ. (A picture is helpful here.)

For example, this captures the idea of what is wrong with the map of the line with the doubled origin over k: we take $\operatorname{Spec} R$ to be the germ of the affine line at the origin, and consider the map of the germ minus the origin to the line with doubled origin. Then we have two choices for how the map can extend over the origin.

2.2. *Exercise.* Make this precise: show that the line with the doubled origin fails the valuative criterion for separatedness.

Proof. (This proof is more telegraphic than I'd like. I may fill it out more later. Because we won't be using this result later in the course, you should feel free to skip it, but you may want to skim it.) One direction is fairly straightforward. Suppose $f : X \to Y$ is separated, and such a diagram (1) were given. suppose g_1 and g_2 were two morphisms

$\operatorname{Spec} R \to X$ making (2) commute. Then $g = (g_1, g_2) : \operatorname{Spec} R \to X \times_Y X$ is a morphism, with $g(\operatorname{Spec} K)$ contained in the diagonal. Hence as $\operatorname{Spec} K$ is dense in $\operatorname{Spec} R$, and g is continuous, $g(\operatorname{Spec} R)$ is contained in the closure of the diagonal. As the diagonal is closed (the separated hypotheses), $g(\operatorname{Spec} R)$ is also contained *set-theoretically* in the diagonal. As $\operatorname{Spec} R$ is reduced, g factors through the reduced induced subscheme structure (§0.1) of the diagonal. Hence g factors through the diagonal:

$$\operatorname{Spec} R \longrightarrow X \xrightarrow{\delta} X \times_Y X,$$

which means $g_1 = g_2$ by Exercise 1.27.

Suppose conversely that f is not separated, i.e. that the diagonal $\Delta \subset X \times_Y X$ is not closed. As $X \times_Y X$ is Noetherian (X is Noetherian, and $X \times_Y X \to X$ is finite type as it is obtained by base change from the finite type $X \to Y$) we have a well-defined notion of dimension of all irreducible closed subsets, and it is bounded. Let P be a point in $\overline{\Delta} - \Delta$ of largest dimension. Let Q be a point in Δ such that $P \in \overline{Q}$. (A picture is handy here.) Let Z be the scheme obtained by giving the reduced induced subscheme structure to $\overline{Q}$. Then P is a codimension 1 point on Z; let $R' = \mathcal{O}_{Z,P}$ be the local ring of Z at P. Then R' is a Noetherian local domain of dimension 1. Let R'' be the normalization of R. Choose any point P'' of $\operatorname{Spec} R''$ mapping to P; such a point exists because the normalization morphism $\operatorname{Spec} R \to \operatorname{Spec} R'$ is surjective (normalization is an integral extension, hence surjective by the Going-up theorem, lecture 21 theorem 1.5). Let R be the localization of R'' at P''. Then R is a normal Noetherian local domain of dimension 1, and hence a discrete valuation ring. Let K be its fraction field. Then $\operatorname{Spec} R \to X \times_Y X$ does not factor through the diagonal, but $\operatorname{Spec} K \to X \times_Y X$ does, and we are done. □

Here is a more general statement. I won't give a proof here, but I think the proof given in Hartshorne Theorem II.4.3 applies (even though the hypotheses are more restrictive).

2.3. *Theorem (Valuative criterion of separatedness). — Suppose $f : X \to Y$ is a quasicompact, quasiseparated morphism. Then f is separated if and only if the following condition holds. For any valuation ring R with function field K, and or any diagram of the form* (1)*, there is at most one morphism* $\operatorname{Spec} R \to X$ *such that the diagram* (2) *commutes.*

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