FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 24

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CONTENTS

Last day: Fibers of morphisms. Properties preserved by base change: open immersions, closed immersions, Segre embedding. Other schemes defined by universal property: reduction, normalization.

Today: normalization (in a field extension), "sheaf Spec**", "sheaf** Proj**", projective morphism.**

1. NORMALIZATION, CONTINUED

Last day, I defined the normalization of a reduced scheme. I have an interesting question for experts: there is a reasonable extension to schemes in general; does anything go wrong? I haven't yet given this much thought, but it seems worth exploring.

I described normalization last day in the case when X is irreducible, and hence integral. In this case of X irreducible, the normalization satisfies the universal property, that if $Y \to X$ is any other dominant morphism from a normal scheme to X, then this morphism factors uniquely through ν:

$$\begin{tikzcd} Y \arrow[rr, dotted, "\exists!"] \arrow[dr] & & \tilde{X} \arrow[dl, "\nu"] \\ & X & \end{tikzcd} .$$

Thus if it exists, then it is unique up to unique isomorphism. We then showed that it exists, using an argument we saw for the third time. (The first time was in the existence of the fibered product. The second was an argument for the existence of the reduction morphism.) The ring-theoretic case got us started: if $X = \operatorname{Spec} R$, then and $\tilde{R}$ is the integral closure of R in its fraction field $\operatorname{Frac}(R)$, then I gave as an exercise that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property.

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1.1. *Exercise.* Show that the normalization morphism is surjective. (Hint: Going-up!)

We now mention some bells and whistles. The following fact is handy.

1.2. *Theorem (finiteness of integral closure). — Suppose* A *is a domain,* $K = \mathrm{Frac}(A)$*,* L/K *is a finite field extension, and* B *is the integral closure of* A *in* L *("the integral closure of* A *in the field extension* L/K*", i.e. those elements of* L *integral over* A*).*
(a) if A *is integrally closed, then* B *is a finitely generated* A*-module.*
(b) if A *is a finitely generated* k*-algebra, then* B *(the integral closure of* A *in its fraction field) is a finitely generated* A*-module.*

I hope to type up a proof of these facts at some point to show you that they are not that bad. Much of part (a) was proved by Greg Brumfiel in 210B last year.

Warning: (b) does *not* hold for Noetherian A in general. I find this very alarming. I don't know an example offhand, but one is given in Eisenbud's book.

1.3. *Exercise.* Show that $\dim \tilde{X} = \dim X$ (hint: see our going-up discussion).

1.4. *Exercise.* Show that if X is an integral finite-type k-scheme, then its normalization $\nu : \tilde{X} \to X$ is a finite morphism.

1.5. *Exercise.* Explain how to generalize the notion of normalization to the case where X is a reduced Noetherian scheme (with possibly more than one component). This basically requires defining a universal property. I'm not sure what the "perfect" definition, but all reasonable universal properties should lead to the same space.

1.6. *Exercise.* Show that if X is an integral finite type k-scheme, then its non-normal points form a closed subset. (This is a bit trickier. Hint: consider where $\nu_* \mathcal{O}_{\tilde{X}}$ has rank 1.) I haven't thought through all the details recently, so I hope I've stated this correctly.

Here is an explicit example to think through some of these ideas.

1.7. Exercise. Suppose $X = \operatorname{Spec} \mathbb{Z}[15i]$. Describe the normalization $\tilde{X} \to X$. (Hint: it isn't hard to find an integral extension of $\mathbb{Z}[15i]$ that is integrally closed. By the above discussion, you've then found the normalization!) Over what points of X is the normalization not an isomorphism?

1.8. *Exercise.* (This is an important generalization!) Suppose X is an integral scheme. Define the *normalization of* X, $\nu : \tilde{X} \to X$, *in a given finite field extension of the function field of* X. Show that $\tilde{X}$ is normal. (This will be hard-wired into your definition.) Show that if either X is itself normal, or X is finite type over a field k, then the normalization in a finite field extension is a finite morphism.

Let's try this in a few cases.

1.9. Exercise. Suppose $X = \operatorname{Spec} \mathbb{Z}$ (with function field $\mathbb{Q}$). Find its integral closure in the field extension $\mathbb{Q}(i)$.

A finite extension K of $\mathbb{Q}$ is called a *number field*, and the integral closure of $\mathbb{Z}$ in K the *ring of integers of* K, denoted $\mathcal{O}_K$. (This notation is a little awkward given our other use of the symbol $\mathcal{O}$.) By the previous exercises, $\operatorname{Spec} \mathcal{O}_K$ is a Noetherian normal domain of dimension 1 (hence regular). This is called a *Dedekind domain*. We think of it as a smooth curve.

1.10. Exercise. (a) Suppose $X = \operatorname{Spec} k[x]$ (with function field $k(x)$). Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Again we get a Dedekind domain.) (b) Suppose $X = \mathbb{P}^1$, with distinguished open $\operatorname{Spec} k[x]$. Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Part (a) involves computing the normalization over one affine open set; now figure out what happens over the "other".)

2. Sheaf Spec

Given an A-algebra, B, we can take its Spec to get an affine scheme over $\operatorname{Spec} A$: $\operatorname{Spec} B \to \operatorname{Spec} A$. I'll now give a universal property description of a globalization of that notation. We will take an arbitrary scheme X, and a quasicoherent sheaf of algebras $\mathcal{A}$ on it. We will define how to take Spec of this sheaf of algebras, and we will get a scheme $\underline{\operatorname{Spec}}\, \mathcal{A} \to X$ that is "affine over X", i.e. the structure morphism is an affine morphism.

We will do this as you might by now expect: for each affine on X, we use our affine construction, and show that everything glues together nicely. We do this instead by describing $\underline{\operatorname{Spec}}\, \mathcal{A} \to X$ in terms of a good universal property: given any morphism $\pi : Y \to X$ along with a morphism of $\mathcal{O}_X$-modules

$$\alpha : \mathcal{A} \to \pi_* \mathcal{O}_Y,$$

there is a unique map $Y \to \underline{\operatorname{Spec}}\, \mathcal{A}$ factoring π, i.e. so that the following diagram commutes,

$$\begin{tikzcd} Y \arrow[rr, dotted, "\exists!"] \arrow[dr, "\pi"'] & & \underline{\operatorname{Spec}}\, \mathcal{A} \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

and an isomorphism $\phi : \mathcal{A} \to \beta_* \mathcal{O}_{\underline{\operatorname{Spec}}\, \mathcal{A}}$ inducing α.

(For experts: we need $\mathcal{O}_X$-modules, and to leave our category of quasicoherent sheaves on X, because we only showed that the pushforward of quasicoherent sheaves are quasicoherent for certain morphisms, where the preimage of each affine was a finite union of affines, the pairwise intersection of which were also finite unions. This notion will soon be formalized as quasicompact and quasiseparated.)

At this point we're getting to be experts on this, so let's show that this $\underline{\mathrm{Spec}}\,\mathcal{A}$ exists. In the case where X is affine, we are done by our affine discussion. In the case where X is quasiaffine, we are done for the same reason as before. And finally, in the case where X is general, we are done once again!

In particular, note that $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$ is an affine morphism.

2.1. *Exercise.* Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \underline{\mathrm{Spec}}\, f_* \mathcal{O}_Z$ of X-schemes.

Hence we can recover any affine morphism in this way. More precisely, a morphism is affine if and only if it is of the form $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$.

2.2. *Exercise ($\underline{\mathrm{Spec}}$ behaves well with respect to base change).* Suppose $f : Z \to X$ is any morphism, and $\mathcal{A}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \underline{\mathrm{Spec}}\,\mathcal{A} \cong \underline{\mathrm{Spec}}\, f^* \mathcal{A}$.

An important example of this $\underline{\mathrm{Spec}}$ construction is the total space of a finite rank locally free sheaf $\mathcal{F}$, which is a *vector bundle*. It is $\underline{\mathrm{Spec}}\,\mathrm{Sym}^* \mathcal{F}^\vee$.

2.3. *Exercise.* Show that this is a vector bundle, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\underline{\mathrm{Spec}}\,\mathrm{Sym}^* \mathcal{F}^\vee|_U \cong \mathbb{A}^n_U$. Show that $\mathcal{F}$ is isomorphic to the sheaf of sections of it.

As an easy example: if $\mathcal{F}$ is a *free* sheaf of rank n, then $\underline{\mathrm{Spec}}\,\mathrm{Sym}^* \mathcal{F}^\vee$ is called $\mathbb{A}^n_X$, generalizing our earlier notions of $\mathbb{A}^n_A$. As the notion of a free sheave behaves well with respect to base change, so does the notion of $\mathbb{A}^n_X$, i.e. given $X \to Y$, $\mathbb{A}^n_Y \times_Y X \cong \mathbb{A}^n_X$.

Here is one last fact that might come in handy.

2.4. *Exercise.* Suppose $f : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\underline{\mathrm{Spec}}\,A$ is "essentially the same as" (=equivalent to) the category of quasicoherent sheaves on X with the structure of $\mathcal{A}$-modules (quasicoherent $\mathcal{A}$-modules on X).

The reason you could imagine caring is when X is quite simple, and $\underline{\mathrm{Spec}}\,\mathcal{A}$ is complicated. We'll use this before long when $X \cong \mathbb{P}^1$, and $\underline{\mathrm{Spec}}\,\mathcal{A}$ is a more complicated curve. (I drew a picture of this.)

3. Sheaf Proj

We'll now do a global (or "sheafy") version of Proj, which we'll denote $\underline{\mathrm{Proj}}$.

Suppose now that $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras of X. To be safe, let me assume that $\mathcal{S}_*$ is locally generated in degree 1 (i.e. there is a cover by small affine open sets, where for each affine open set, the corresponding algebra is generated in degree 1), and $\mathcal{S}_1$ is finite type. We will define $\underline{\mathrm{Proj}}\,\mathcal{S}_*$.

The essential ideal is that we do this affine by affine, and then glue the result together. But as before, this is tricky to do, but easier if you state the right universal property.

As a preliminary, let me re-examine our earlier theorem, that "Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of $\mathcal{O}_X^*$."

I will now describe this in a more "relative" setting, where relative means that we do this with morphisms of schemes. We begin with a relative notion of base-point free. Suppose $f : Y \to X$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on Y. We say that $\mathcal{L}$ is *relatively base point free* if for every point $p \in X$, $q \in Y$, with $f(q) = p$, there is a neighborhood U for which there is a section of $\mathcal{L}$ over $f^{-1}(U)$ not vanishing at q. Similarly, we define *relatively generated by global sections* if there is a neighborhood U for which there are sections of $\mathcal{L}$ over $f^{-1}(U)$ generating *every* stalk of $f^{-1}(U)$. This is admittedly hideous terminology. (One can also define *relatively generated by global sections at a point* $p \in Y$. See class 16 where we defined these notions in a non-relative setting. In class 32, this will come up again.) More generally, we can define the notion of "relatively generated by global sections by a subsheaf of $f_*\mathcal{L}$".

Definition. $(\underline{\mathrm{Proj}}\,\mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{S}_*}(1)) \to X$ satisfies the following universal property. Given any other X-scheme Y with an invertible sheaf $\mathcal{L}$, and a map of graded $\mathcal{O}_X$-algebras

$$\alpha : \mathcal{S}_* \to \oplus_{n=0}\pi_*\mathcal{L}^{\otimes n},$$

such that $\mathcal{L}$ is relatively generated by the global sections of $\alpha(S_1)$, there is a unique factorization

$$\begin{tikzcd} Y \arrow[rr, dotted, "\exists! f"] \arrow[dr, "\pi"'] & & \underline{\mathrm{Proj}}\,\mathcal{S}_* \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

and a canonical isomorphism $\mathcal{L} \cong f^*\mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{S}_*}(1)$ and a morphism $\mathcal{S}_* \to \oplus_n \beta_*\mathcal{O}(n)$ inducing α.

In particular, $\underline{\mathrm{Proj}}\,\mathcal{S}_*$ comes with an invertible sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{S}_*}(1)$, and this $\mathcal{O}(1)$ should be seen as part of the data.

This definition takes some getting used to.

But we prove this as usual!

We first deal with the case where X is affine, say $X = \mathrm{Spec}\,A$, $\mathcal{S}_* = \tilde{S}_*$. You won't be surprised to hear that in this case, $(\mathrm{Proj}\,S_*, \mathcal{O}(1))$ satisfies the universal property.

We outline why. Clearly, given a map $Y \to \mathrm{Proj}\,S_*$, we get a pullback map α. Conversely, given such a pullback map, we want to show that this induces a (unique) map $Y \to \mathrm{Proj}\,S_*$. Now because S_* is generated in degree 1, we have a closed immersion

$\operatorname{Proj} S_* \hookrightarrow \operatorname{Proj} \operatorname{Sym}^* S_1$. The map in degree 1, $S_1 \to \pi_* \mathcal{L}$, gives a map $Y \to \operatorname{Proj} \operatorname{Sym}^* S_1$ by our magic theorem "Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of $\mathcal{O}_X^*$."

3.1. Exercise. Complete this argument that if $X = \operatorname{Spec} A$, then $(\operatorname{Proj} \mathcal{S}_*, \mathcal{O}(1))$ satisfies the universal property.

3.2. Exercise. Show that $(\operatorname{Proj} \mathcal{S}_*, \mathcal{O}(1))$ exists in general, by following the analogous universal property argument: show that it exists for X quasiaffine, then in general.

3.3. *Exercise ($\underline{\operatorname{Proj}}$ behaves well with respect to base change).* Suppose $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\operatorname{Proj}}\, \mathcal{S}_*$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\underline{\operatorname{Proj}}\, f^*\mathcal{S}_*, \mathcal{O}_{\underline{\operatorname{Proj}}\, f^*\mathcal{S}_*}(1)) \cong (Y \times_X \underline{\operatorname{Proj}}\, \mathcal{S}_*, g^*\mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)) \cong$$

where g is the natural morphism in the base change diagram

$$\begin{tikzcd} Y \times_X \underline{\operatorname{Proj}}\, \mathcal{S}_* \arrow[r, "g"] \arrow[d] & \underline{\operatorname{Proj}}\, \mathcal{S}_* \arrow[d] \\ Y \arrow[r] & X. \end{tikzcd}$$

3.4. *Definition.* If $\mathcal{F}$ is a finite rank locally free sheaf on X. Then $\underline{\operatorname{Proj}} \operatorname{Sym}^* \mathcal{F}$ is called its *projectivization*. If $\mathcal{F}$ is a free sheaf of rank $n+1$, then we define $\mathbb{P}_X^n := \underline{\operatorname{Proj}} \operatorname{Sym}^* \mathcal{F}$. (Then $\mathbb{P}^n_{\operatorname{Spec} A}$ agrees with our earlier definition of $\mathbb{P}^n_A$.) Clearly this notion behaves well with respect to base change.

This "relative $\mathcal{O}(1)$" we have constructed is a little subtle. Here are couple of exercises to give you practice with the concept.

3.5. *Exercise.* $\underline{\operatorname{Proj}}(\mathcal{S}_*[t]) \cong \underline{\operatorname{Spec}}\, \mathcal{S}_* \coprod \underline{\operatorname{Proj}}\, \mathcal{S}_*$, where $\underline{\operatorname{Spec}}\, \mathcal{S}_*$ is an open subscheme, and $\underline{\operatorname{Proj}}\, \mathcal{S}_*$ is a closed subscheme. Show that $\underline{\operatorname{Proj}}\, \mathcal{S}*$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\underline{\operatorname{Proj}}\, N}(1)$. (This is the generalization of the projective and affine cone. At some point I should give an explicit reference to our earlier exercise on this.)

3.6. *Exercise.* Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\operatorname{Proj}}\, \mathcal{S}_*$ to exist. Define $\mathcal{S}'_* = \oplus_{n=0} \mathcal{S}_n \otimes \mathcal{L}_n$. Give a natural isomorphism of X-schemes

$$(\underline{\operatorname{Proj}}\, \mathcal{S}'_*, \mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}'_*}(1)) \cong (\underline{\operatorname{Proj}}\, \mathcal{S}_*, \mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1) \otimes \pi^*\mathcal{L}),$$

where $\pi : \underline{\operatorname{Proj}}\, \mathcal{S}_* \to X$ is the structure morphism. In other words, informally speaking, the $\underline{\operatorname{Proj}}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$.

3.7. Projective morphisms.

If you are tuning out because of these technicalities, please tune back in! I now want to define an essential notion.

Recall that we have recast affine morphisms in the following way: $X \to Y$ is an affine morphism if $X \cong \underline{\operatorname{Spec}}\, \mathcal{A}$ for some quasicoherent sheaf of algebras $\mathcal{A}$ on Y.

I will now *define* the notion of a projective morphism similarly.

3.8. *Definition.* A morphism $X \to Y$ is *projective* if there is an isomorphism

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr] & & \underline{\operatorname{Proj}}\, \mathcal{S}_* \arrow[dl] \\ & Y & \end{tikzcd}$$

for a quasicoherent sheaf of algebras $\mathcal{S}_*$ on Y satisfying the required hypothesis for $\underline{\operatorname{Proj}}$ to exist.

Two warnings! 1. Notice that I didn't say anything about the $\mathcal{O}(1)$, which is an important definition. The notion of affine morphism is affine-local on the target, but this notion is not affine-local on the target! (In nice circumstances it is, as we'll see later. We'll also see an example where this is not.) 2. Hartshorne gives a different definition; I'm following the more general definition of Grothendieck. But again, these definitions turn out to be the same in nice circumstances.

This is the "relative version" of $\operatorname{Proj} S_* \to \operatorname{Spec} A$.

3.9. *Exercise.* Show that closed immersions are projective morphisms. (Hint: Suppose the closed immersion $X \to Y$ corresponds to $\mathcal{O}_Y \to \mathcal{O}_X$. Consider $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_i = \mathcal{O}_Y$ for $i > 1$.)

3.10. *Exercise (suggested by Kirsten).* Suppose $f : X \hookrightarrow \mathbb{P}^n_S$ where S is some scheme. Show that the structure morphism $\pi : X \to S$ is a projective morphism as follows: let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n_S}(1)$, and show that $X = \underline{\operatorname{Proj}}\, \pi_*\mathcal{L}^{\otimes n}$.

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