FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 23

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CONTENTS

Last day: Morphisms to (quasi)projective schemes, and invertible sheaves; fibered products.

Today: Fibers of morphisms. Properties preserved by base change: open immersions, closed immersions, Segre embedding. Other schemes defined by universal property: reduction, normalization.

Last day, I showed you that fibered products exist, and I gave an argument that had fairly few moving parts: fibered products exist when the schemes in question are affine schemes; the universal property; and the fact that morphisms glue. I'll give you an exercise later today to give you a chance to make a similar argument, when I give the universal property for reducedness.

1. FIBERS OF MORPHISMS

We can informally interpret fibered product in the following geometric way. Suppose $Y \to Z$ is a morphism. We interpret this as a "family of schemes parametrized by a base scheme (or just plain *base*) Z." Then if we have another morphism $X \to Z$, we interpret the induced map $X \times_Z Y \to X$ as the "pulled back family". I drew a picture of this on the blackboard. I discussed the example: the family $y^2 z = x^3 + txz^2$ of cubics in $\mathbb{P}^2$ parametrized by the affine line, and what happens if you pull back to the affine plane via $t = uv$, to get the family $y^2 z = x^3 + uvxz^2$.

For this reason, fibered product is often called *base change* or *change of base* or *pullback*.

For instance, if X is a closed point of Z, then we will get the fiber over Z. As an example, consider the map of schemes $f : Y = \operatorname{Spec} \mathbb{Q}[t] \to Z = \operatorname{Spec} \mathbb{Q}[u]$ given by $u \mapsto t^2$ (or

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$u = t^2$). (I drew a picture on the blackboard. It looked like a parabola with horizontal axis of symmetry, projecting to the x-axis.) The fiber above $u = 1$ corresponds to the base change $X = \operatorname{Spec} \mathbb{Q}[u]/(u-1) \to \operatorname{Spec} \mathbb{Q}[u]$. Let's do the algebra: $X \times_Z Y = \operatorname{Spec} \mathbb{Q}[t,u]/(u-1, u-t^2) \cong \operatorname{Spec} \mathbb{Q}[t]/(t^2-1) \cong \operatorname{Spec} \mathbb{Q}[t]/(t-1) \times \mathbb{Q}[t]/(t+1)$. We see two reduced points (at "$u = 1, t = 1$ and $u = 1, t = -1$").

Next let's examine the fiber above $u = 0$. We get $\operatorname{Spec} \mathbb{Q}[t]/(t^2)$ — a point with non-reduced structure!

Finally, let's consider $u = -1$. We get $\operatorname{Spec} \mathbb{Q}[t]/(t^2+1)$. We get a single reduced point. The residue field $\mathbb{Q}(i)$ is a degree 2 field extension over $\mathbb{Q}$.

(Notice that in each case, we get something of "size two", informally speaking. One way of making this precise is that the rank of the sheaf $f_* \mathcal{O}_Y$ is rank 2 everywhere. In the first case, we see it as getting two different points. In the second, we get one point, with non-reduced behavior. In the last case, we get one point, of "size two". We will later see this "constant rank of $f_* \mathcal{O}_Y$" as symptomatic of the fact that this morphism is "particularly nice", i.e. finite and flat.)

We needn't look at fibers over just closed points; we can consider fibers over any points. More precisely, if p is a point of Z with residue field K, then we get a map $\operatorname{Spec} K \to Z$, and we can base change with respect to this morphism.

In the case of the generic point of $\operatorname{Spec} \mathbb{Q}[u]$ in the above example, we have $K = \mathbb{Q}(u)$, and $\mathbb{Q}[u] \to \mathbb{Q}(u)$ is the inclusion of the generic point. Let $X = \operatorname{Spec} \mathbb{Q}(u)$. Then you can verify that $X \times_Z Y = \operatorname{Spec} \mathbb{Q}[t,u]/(u-t^2) \otimes \mathbb{Q}(u) \cong \operatorname{Spec} \mathbb{Q}(t)$. We get the morphism $\mathbb{Q}(u) \to \mathbb{Q}(t)$ given by $u = t^2$ — a quadratic field extension.

Implicit here is a notion I should make explicit, about how you base change with respect to localization. Given $A \to B$, and a multiplicative set S of A, we have $(S^{-1}A) \otimes_A B \cong S^{-1}B$, where $S^{-1}B$ has the obvious interpretation. In other words,

$$\begin{tikzcd} S^{-1}B & B \arrow[l] \\ S^{-1}A \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

is "cofiber square" (or "pushout diagram").

1.1. *Remark: Geometric points.* We have already given two meanings for the "points of a scheme". We used one to define the notion of a scheme. Secondly, if T is a scheme, people sometimes say that $\operatorname{Hom}(T, X)$ are the "T-valued points of X". That's already confusing. But also, people say that the geometric points correspond to $\operatorname{Hom}(T, X)$ where T is the Spec of an algebraically closed field. Then for example the *geometric fibers* are the fibers over geometric points. In the example above, here is a geometric point: $\operatorname{Spec} \overline{\mathbb{Q}}[u]/(u-1) \to \operatorname{Spec} \mathbb{Q}[u]$. And here is a geometric fiber: $\operatorname{Spec} \overline{\mathbb{Q}}[t]/(t^2-1)$. Notice that the geometric fiber above $u = -1$ also consists of two points, unlike the "usual" fiber.

(I should check: possibly the definition should just be for T the algebraic closure of the residue field of a not-necessarily-closed point.)

1.2. *Exercise for the arithmetically-minded.* Show that for the morphism $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, all geometric fibers consist of two reduced points. This exercise should be removed if I have the wrong definition of geometric point!

We will discuss more about geometric points and properties of geometric fibers shortly.

2. Properties preserved by base change

We now discuss a number of properties that behave well under base change.

We've already shown that the notion of "open immersion" is preserved by base change (problem 6 on problem set 9, see class 19). We did this by explicitly describing what the fibered product of an open immersion is: if $Y \hookrightarrow Z$ is an open immersion, and $f : X \to Z$ is any morphism, then we checked that the open subscheme $f^{-1}(Y)$ of X satisfies the universal property of fibered products.

2.1. *Important exercise (problem 8+ on the last problem set).* Show that the notion of "closed immersion" is preserved by base change. (This was stated in class 19.) Somewhat more precisely, given a fiber diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

where $Y \hookrightarrow Z$ is a closed immersion, then $W \hookrightarrow X$ is as well. (Hint: in the case of affine schemes, you have done this before in a different guise — see problem B3 on problem set 1!) In the course of the proof, you will show that W is cut out by the same equations in X as Y is in Z, or more precisely by pullback of those equations. Hence fibered products (over k) of schemes of finite type over k may be computed easily:

$$\operatorname{Spec} k[x_1,\dots,x_m]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m))\times_{\operatorname{Spec} k}$$

$$\operatorname{Spec} k[y_1,\dots,y_m]/(g_1(y_1,\dots,y_m),\dots,g_s(y_1,\dots,y_m))$$

$$\cong \operatorname{Spec} k[x_1,\dots,x_m,y_1,\dots,y_m]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m),$$

$$g_1(y_1,\dots,y_m),\dots,g_s(y_1,\dots,y_m)).$$

We sometimes say that W is the *scheme-theoretic pullback of* Y, *scheme-theoretic inverse image*, or *inverse image scheme of* Y. The ideal sheaf of W is sometimes called the *inverse image (quasicoherent) ideal sheaf*.

Note for experts: It is not necessarily the quasicoherent pullback (f^*) of the ideal sheaf, as the following example shows. (Thanks Joe!)

$$\begin{tikzcd} \operatorname{Spec} k[x]/(x) \arrow[r] \arrow[d] & \operatorname{Spec} k[x]/(x) \arrow[d] \\ \operatorname{Spec} k[x]/(x) \arrow[r] & \operatorname{Spec} k[x] \end{tikzcd}$$

Instead, the correct thing to pullback (the thing that "pulls back well") is the surjection $\mathcal{O}_Z \to \mathcal{O}_Y \to 0$, which pulls back to $\mathcal{O}_X \to \mathcal{O}_W \to 0$. The key issue is that pullback of quasicoherent sheaves is right-exact, so we shouldn't expect the pullback of $0 \to \mathcal{I}_{Y/Z} \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0$ to be exact, only right-exact. (Thus for example we get a natural map $f^*\mathcal{I}_{Y/Z} \to \mathcal{I}_{W/X}$.)

Similarly, other important properties are preserved by base change.

2.2. *Exercise.* Show that the notion of "morphism locally of finite type" is preserved by base change. Show that the notion of "affine morphism" is preserved by base change. Show that the notion of "finite morphism" is preserved by base change.

2.3. *Exercise.* Show that the notion of "quasicompact morphism" is preserved by base change.

2.4. *Exercise.* Show that the notion of "morphism of finite type" is preserved by base change.

2.5. *Exercise.* Show that the notion of "quasifinite morphism" (= finite type + finite fibers) is preserved by base change. (Note: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)

2.6. *Exercise.* Show that surjectivity is preserved by base change (or fibered product). In other words, if $X \to Y$ is a surjective morphism, then for any $Z \to Y$, $X \times_Y Z \to Z$ is surjective. (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

2.7. *Exercise.* Show that the notion of "irreducible" is not necessarily preserved by base change. Show that the notion of "connected" is not necessarily preserved by base change. (Hint: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[i]$.)

If X is a scheme over a field k, it is said to be *geometrically irreducible* if its base change to $\overline{k}$ (i.e. $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$) is irreducible. Similarly, it is *geometrically connected* if its base change to $\overline{k}$ (i.e. $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$) is connected. Similarly also for *geometrically reduced* and *geometrically integral*. We say that $f : X \to Y$ has *geometrically irreducible (resp. connected, reduced, integral) fibers* if the geometric fibers are geometrically irreducible (resp. connected, reduced, integral).

If you care about such notions, see Hartshorne Exercise II.3.15 for some facts (stated in a special case). In particular, to check geometric irreducibility, it suffices to check over *separably closed* (not necessarily algebraically closed) fields. To check geometric reducedness, it suffices to check over *perfect* fields.

2.8. *Exercise.* Show that $\operatorname{Spec} \mathbb{C}$ is not a geometrically irreducible $\mathbb{R}$-scheme. If $\operatorname{char} k = p$, show that $\operatorname{Spec} k(u)$ is not a geometrically reduced $\operatorname{Spec} k(u^p)$-scheme.

2.9. *Exercise.* Show that the notion of geometrically irreducible (resp. connected, reduced, integral) fibers behaves well with respect to base change.

On a related note:

2.10. *Exercise (less important).* Suppose that l/k is a finite field extension. Show that a k-scheme X is normal if and only if $X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ is normal. Hence deduce that if k is any field, then $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal. (I think this was promised earlier.) Hint: we showed earlier (Problem B4 on set 4) that $\operatorname{Spec} k[a, b, c, d]/(a^2 + b^2 + c^2 + d^2)$ is normal.

3. Products of projective schemes: The Segre embedding

I will next describe products of projective A-schemes over A. The case of greatest initial interest is if $A = k$. (A reminder of why we like projective schemes. (i) it is an easy way of getting interesting non-affine schemes. (ii) we get lots of schemes of classical interest. (iii) we have a hard time thinking of anything that isn't projective or an open subset of a projective. (iv) a k-scheme is a first approximation of what we mean by compact.)

In order to do this, I need only describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, because any projective scheme has a closed immersion in some $\mathbb{P}^m_A$, and closed immersions behave well under base change: so if $X \hookrightarrow \mathbb{P}^m_A$ and $Y \hookrightarrow \mathbb{P}^n_A$ are closed immersions, then $X \times_A Y \hookrightarrow \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ is also a closed immersion, cut out by the equations of X and Y.

We'll describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, and see that it too is a projective A-scheme. Consider the map $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$ given by

$$([x_0; \dots; x_m], [y_0; \dots; y_n]) \to [z_{00}; z_{01}; \cdots; z_{ij}; \cdots; z_{mn}] = [x_0 y_0; x_0 y_1; \cdots; x_i y_j; \cdots x_m y_n].$$

First, you should verify that this is a well-defined morphism! On the open chart $U_i \times V_j$, this gives a map $(x_{0/i}, \dots, x_{m/i}, y_{0/j}, \dots, y_{n/j}) \mapsto [x_{0/i} y_{0/j}; \dots; x_{i/i} y_{j/j}; \dots; x_{m/i} y_{n/j}]$. Note that this gives an honest map to projective space — not all the entries on the right are zero, as one of the entries $(x_{i/i} y_{j/j})$ is 1.

(Aside: we now well know that a map to projective space corresponds to an invertible sheaf with a bunch of sections. The invertible sheaf on this case is $\pi_1^* \mathcal{O}_{\mathbb{P}^m_A}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n_A}(1)$, where π_i are the projections of the product onto the two factors. The notion $\boxtimes$ is often used for this notion, when you pull back sheaves from each factor of a product, and tensor. For example, this invertible sheaf could be written $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$. People often write $\mathcal{O}(a) \boxtimes \mathcal{O}(b)$ for $\mathcal{O}(a, b)$.)

I claim this morphism is a closed immersion. (We are essentially using Exercise 3.2 in the class 21 notes, problem 40 in problem set 9. But don't waste your time by looking back at it.) Let's check this on the open set where $z_{ab} \neq 0$. Without loss of generality, I'll take $a = b = 0$, to make notation simpler. Then the preimage of this open set in $\mathbb{P}^m_A \times \mathbb{P}^n_A$ is the locus where $x_0 \neq 0$ and $y_0 \neq 0$, i.e. $U_0 \times V_0$, U_0 and V_0 are the usual distinguished open sets of $\mathbb{P}^m_A$ and $\mathbb{P}^n_A$ respectively. The coordinates here are $x_{1/0}, \ldots, x_{m/0}, y_{1/0}, \ldots, y_{n/0}$. Thus the map corresponds to $z_{ab/00} \to x_{a/0} y_{b/0}$, which clearly induces a surjection of rings

$$A[z_{00/00}, \ldots, z_{mn/00}] \to A[x_{1/0}, \ldots, x_{m/0}, y_{1/0}, \ldots, y_{n/0}].$$

(Recall that $z_{a0/00} \mapsto x_{a/0}$ and $z_{0b/00} \mapsto y_{b/0}$.)

Hence we are done! This map is called the *Segre morphism* or *Segre embedding*. If A is a field, the image is called the *Segre variety* — although we don't yet know what a variety is!

Here are some useful comments.

3.1. *Exercise.* Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank} \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

3.2. *Example.* Let's consider the first non-trivial example, when $m = n = 1$. We get $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. We get a single equation

$$\operatorname{rank} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 1,$$

i.e. $a_{00} a_{11} - a_{01} a_{10} = 0$. We get our old friend, the quadric surface! Hence: the nonsingular quadric surface $wz - xy = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that we can reinterpret the rulings; I pointed this out on the model. Since (by diagonalizability of quadratics) all nonsingular quadratics over an algebraically closed field are isomorphic, we have that all nonsingular quadric surfaces over an algebraically closed field are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that this is not true even over a field that is not algebraically closed. For example, over $\mathbb{R}$, $w^2 + x^2 + y^2 + z^2 = 0$ is not isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$. Reason: the former has no real points, while the latter has lots of real points.

3.3. Let's return to the general Segre situation. We can describe the closed subscheme alternatively the Proj of the subring R of

$$A[x_0, \dots, x_m, y_0, \dots, y_n]$$

generated by monomials of equal degree in the x's and the y's. Using this, you can give a co-ordinate free description of this product (i.e. without using the co-ordinates x_i and y_j): $\mathbb{P}^m_A \times_A \mathbb{P}^n_A = \operatorname{Proj} R$ where

$$R = \oplus_{i=0}^{\infty} \operatorname{Sym}^i H^0(\mathbb{P}^m_A, \mathcal{O}(1)) \otimes \operatorname{Sym}^i H^0(\mathbb{P}^n_A, \mathcal{O}(1)).$$

Kirsten asks an interesting question: show that $\mathcal{O}(a, b)$ gives a closed immersion to projective space if $a, b > 0$.

You may want to ponder how to think of products of three projective spaces.

4. Other schemes defined by universal property: reduction, normalization

I now want to define other schemes using universal properties, in ways that are vaguely analogous to fibered product.

As a warm-up, I'd like to revisit an earlier topic: reduction of a scheme. Recall that if X is a scheme, we defined a closed immersion $X^{\text{red}} \hookrightarrow X$. (See the comment just before §1.4 in class 19.) I'd like to revisit this.

4.1. *Potentially enlightening exercise.* Show that $X^{\text{red}} \to X$ satisfies the following universal property: any morphism from a reduced scheme Y to X factors uniquely through X^{red}.

$$\begin{tikzcd} Y \arrow[rr, dotted, "\exists!"] \arrow[dr] & & X^{\text{red}} \arrow[dl] \\ & X & \end{tikzcd} .$$

You can use this as a definition for $X^{\text{red}} \to X$. Let me walk you through part of this. First, prove this for X affine. (Here you use the fact that we know that maps to an affine scheme correspond to a maps of global sections in the other direction.) Then use the universal property to show the result for quasiaffine X. Then use the universal property to show it in general. **Oops! I don't think I've defined quasiaffine before. It is any scheme that can be expressed as an open subset of an affine scheme. I should eventually put this definition earlier in the course notes, but may not get a chance to. It may appear in the class 22 notes, which are yet to be written up. The concept is reintroduced yet again in Exercise 4.4 below.**

4.2. Normalization.

I now want to tell you how to normalize a reduced Noetherian scheme. A normalization of a scheme X is a morphism $\nu : \tilde{X} \to X$ from a normal scheme, where ν induces a bijection of components of $\tilde{X}$ and X, and ν gives a birational morphism on each of the components; it will be nicer still, as it will satisfy a universal property. (I drew a picture of a normalization of a curve.) **Oops! I didn't define *birational* until class 27. Please just plow ahead! I may later patch this anachronism, but most likely I won't get the chance.**

I'll begin by dealing with the case where X is irreducible, and hence integral. (I'll then deal with the more general case, and also discuss normalization in a function field extension.)

In this case of X irreducible, the normalization satisfies dominant morphism from an irreducible normal scheme to X, then this morphism factors uniquely through ν:

$$\begin{tikzcd} Y \arrow[rr, dotted, "\exists!"] \arrow[dr] & & \tilde{X} \arrow[dl, "\nu"] \\ & X & \end{tikzcd} .$$

Thus if it exists, then it is unique up to unique isomorphism. We now have to show that it exists, and we do this in the usual way. We deal first with the case where X is affine, say $X = \operatorname{Spec} R$, where R is an integral domain. Then let $\tilde{R}$ be the integral closure of R in its fraction field $\operatorname{Frac}(R)$.

4.3. *Exercise.* Show that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property.

4.4. *Exercise.* Show that normalizations exist for any quasiaffine X (i.e. any X that can be expressed as an open subset of an affine scheme).

4.5. *Exercise.* Show that normalizations exist in general.

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