FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 22

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CONTENTS

Last day: integral extensions, Going-up theorem, Noether Normalization, proof that transcendence degree = Krull dimension, proof of Chevalley's theorem.

Today: Morphisms to (quasi)projective schemes, and invertible sheaves; fibered products; fibers.

1. IMPORTANT EXAMPLE: MORPHISMS TO PROJECTIVE (AND QUASIPROJECTIVE) SCHEMES, AND INVERTIBLE SHEAVES

1.1. *Important theorem. — Maps to* $\mathbb{P}^n$ *correspond to* $n+1$ *sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of* $\mathcal{O}_X^*$.

Here more precisely is the correspondence. If you have $n+1$ sections, then away from the intersection of their zero-sets, we have a morphism. Conversely, if you have a map to projective space $f : X \to \mathbb{P}^n$, then we have $n+1$ sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, corresponding to the hyperplane sections, $x_0, \dots, x_{n+1}$. then $f^*x_0, \dots, f^*x_{n+1}$ are sections of $f^*\mathcal{O}_{\mathbb{P}^n}(1)$, and they have no common zero.

So to prove this, we just need to show that these two constructions compose to give the identity in either direction.

Given $n+1$ sections $s_0, \dots, s_n$ of an invertible sheaf. We get trivializations on the open sets where each one vanishes. The transition functions are precisely s_i/s_j on $U_i \cap U_j$. We pull back $\mathcal{O}(1)$ by this map to projective space, This is trivial on the distinguished open sets. Furthermore, $f^*D(x_i) = D(s_i)$. Moreover, $s_i/s_j = f^*x_i/x_j$. Thus starting with the

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$n+1$ sections, taking the map to the projective space, and pulling back $\mathcal{O}(1)$ and taking the sections $x_0, \dots, x_n$, we recover the s_i's. That's one of the two directions.

Correspondingly, given a map $f : X \to \mathbb{P}^n$, let $s_i = f^* x_i$. The map $[s_0; \cdots ; s_n]$ is precisely the map f. We see this as follows. The preimage of U_i is $D(s_i) = D(f^* x_i) = f^* D(x_i)$. So the right open sets go to the right open sets. And $D(s_i) \to D(x_i)$ is precisely by $s_j/s_i = f^*(x_j/x_i)$. $\square$

1.2. *Exercise (Automorphisms of projective space).* Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1) \times (n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

This exercise will be useful later, especially for the case $n = 1$.

(A question for experts: why did I not state that previous exercise over an arbitrary base ring A? Where does the argument go wrong in that case?)

1.3. *Neat Exercise.* Show that any map from projective space to a smaller projective space is constant.

Here are some useful phrases to know.

A *linear series* on a scheme X over a field k is an invertible sheaf $\mathcal{L}$ and a finite-dimensional k-vector space V of sections. (We will not require that this vector space be a subspace of $\Gamma(X, \mathcal{L})$; in general, we just have a map $V \to \Gamma(X, \mathcal{L})$.) If the linear series is $\Gamma(X, \mathcal{L})$, we call it a *complete linear series*, and is often written $|\mathcal{L}|$. Given a linear series, any point $x \in X$ on which all elements of the linear series V vanish, we say that x is a *base-point* of V. If V has no base-points, we say that it is *base-point-free*. The union of base-points is called the *base locus*. In fact, it naturally has a scheme-structure — it is the (scheme-theoretic) intersection of the vanishing loci of the elements of V (or equivalently, of a basis of V). In this incarnation, it is called the *base scheme* of the linear series.

Then Theorem 1.1 says that each base-point-free linear series gives a morphism to projective space $X \to \mathbb{P}V^* = \operatorname{Proj} \oplus_n \mathcal{L}^{\otimes n}$. The resulting morphism is often written $X \xrightarrow{|V|} \mathbb{P}^n$. (I may not have this notation quite standard; I should check with someone. I always forget whether I should use "linear system" or "linear series".)

1.4. *Exercise.* If the image scheme-theoretically lies in a hyperplane of projective space, we say that it is *degenerate* (and otherwise, *non-degenerate*). Show that a base-point-free linear series V with invertible sheaf $\mathcal{L}$ is non-degenerate if and only if the map $V \to \Gamma(X, \mathcal{L})$ is an inclusion. Hence in particular a complete linear series is always non-degenerate.

Example: The Veronese and Segre morphisms. *Whoops! We don't know much about fibered products yet, so the Segre discussion may be a bit confusing. But fibered products are* *coming very very shortly...* The Veronese morphism can be interpreted in this way. The dth Veronese morphism on $\mathbb{P}^n$ corresponds to the complete linear series $|\mathcal{O}_{\mathbb{P}^n}(d)|$.

The Segre morphism can also be interpreted in this way. In case I haven't defined it yet, suppose $\mathcal{F}$ is a quasicoherent sheaf on a Z-scheme X, and $\mathcal{G}$ is a quasicoherent sheaf on a Z-scheme Y. Let π_X, π_Y be the projections from $X \times_Z Y$ to X and Y respectively. Then $\mathcal{F} \boxtimes \mathcal{G}$ is defined to be $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$. In particular, $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a, b)$ is defined to be $\mathcal{O}_{\mathbb{P}^m}(a) \boxtimes \mathcal{O}_{\mathbb{P}^n}(b)$ (over any base Z). The Segre morphism $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ corresponds to the complete linear system for the invertible sheaf $\mathcal{O}(1,1)$.

Both of these complete linear systems are easily seen to be base-point-free (*exercise*). We still have to check by hand that they are closed immersions. (We will later see, in class 34, a criterion for linear series to be a closed immersion, at least in the special case where we are working over an algebraically closed field.)

2. Fibered products

We will now construct the fibered product in the category of schemes. In other words, given $X, Y \to Z$, we will show that $X \times_Z Y$ exists. (Recall that the *absolute product* in a category is the fibered product over the final object, so $X \times Y = X \times_{\mathbb{Z}} Y$ in the category of schemes, and $X \times Y = X \times_S Y$ if we are implicitly working in the category of S-schemes, for example if S is the spectrum of a field.)

Here is a notation warning: in the literature (and indeed in this class) lazy people wanting to save chalk and ink will write $\times_k$ for $\times_{\operatorname{Spec} k}$, and similarly for $\times_{\mathbb{Z}}$. In fact it already happened in the paragraph above!

As always when showing that certain objects defined by universal properties exist, we have two ways of looking at the objects in practice: by using the universal property, or by using the details of the construction.

The key idea, roughly, is this: we cut everything up into affine open sets, do fibered products in that category (where it turns out we have seen the concept before in a different guise), and show that everything glues nicely. We can't do this too naively (e.g. by induction), as in general we won't be able to cut things into a finite number of affine open sets, so there will be a tiny bit of cleverness.

The argument will be an inspired bit of abstract nonsense, where we'll have to check almost nothing. This sort of argument is very powerful, and we will use it immediately after to construct lots of other interesting notions, so please pay attention!

Before we get started, here is a sign that something interesting happens for fibered products of schemes. Certainly you should believe that if we take the product of two affine lines (over your favorite algebraically field k, say), you should get the affine plane: $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ should be $\mathbb{A}^2_k$. But the underlying set of the latter is *not* the underlying set of the former —- we get additional points! I'll give an exercise later for you to verify this.

Let's take a break to introduce some language. Say

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is a *fiber diagram* or *Cartesian diagram* or *base change diagram*. It is often called a *pullback diagram*, and $W \to X$ is called the *pullback* of $Y \to Z$ by f, and W is called the *pullback* of Y by f.

At this point, I drew some pictures on the blackboard giving some intuitive idea of what a pullback does. If $Y \to Z$ is a "family of schemes", then $W \to Z$ is the "pulled back family". To make this more explicit or precise, I need to tell you about fibers of a morphism. I also want to give you a bunch of examples. But before doing either of these things, I want to tell you how to compute fibered products in practice.

Okay, lets get to work.

2.1. *Theorem (fibered products always exist). — Suppose* $f : X \to Z$ *and* $g : Y \to Z$ *are morphisms of schemes. Then the fibered product*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

exists in the category of schemes.

We have an extended proof by universal property.

First, if X, Y, Z are affine schemes, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, the fibered product exists, and is $\operatorname{Spec} A \otimes_C B$. Here's why. Suppose W is any scheme, along with morphisms $f'' : W \to X$ and $g'' : W \to Y$ such that $f \circ f'' = g \circ g''$ as morphisms $W \to Z$. We hope that there exists a unique $h : W \to \operatorname{Spec} A \otimes_C B$ such that $f'' = g' \circ h$ and $g'' = f' \circ h$.

$$\begin{tikzcd} W \arrow[dr, "\exists!?"] \arrow[drr, "g''"] \arrow[ddr, "f''"'] & & \\ & \operatorname{Spec} A \otimes_C B \arrow[r, "f'"'] \arrow[d, "g'"] & \operatorname{Spec} B \arrow[d, "g"] \\ & \operatorname{Spec} A \arrow[r, "f"] & \operatorname{Spec} C \end{tikzcd}$$

But maps to affine schemes correspond precisely to maps of global sections in the other direction (class 19 exercise 0.1):

$$\begin{tikzcd}
\Gamma(W, \mathcal{O}_W) & & \\
& A \otimes_C B \arrow[ul, "\exists!?"'] & B \arrow[l, "f'"] \arrow[ull, "g''"'] \\
& A \arrow[u, "g'"] \arrow[uul, "f''"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

But this is precisely the universal property for tensor product! (The tensor product is the cofibered product in the category of rings.)

Thus indeed $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, and more generally $(\mathbb{A}^1)^n \cong \mathbb{A}^n$.

Exercise. Show that the fibered product does not induce a bijection of points

$$\text{points}(\mathbb{A}^1_k) \times \text{points}(\mathbb{A}^1_k) \longrightarrow \text{points}(\mathbb{A}^2_k).$$

Thus products of schemes do something a little subtle on the level of sets.

Second, we note that the fibered product with open immersions always exists: if $Y \hookrightarrow Z$ an open immersion, then for any $f : X \to Z$, $X \times_Z Y$ is the open subset $f^{-1}(Y)$. (More precisely, this open subset satisfies the universal property.) We proved this in class 19 (exercise 1.2).

$$\begin{tikzcd}
f^{-1}(Y) \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

(An exercise to give you practice with this concept: show that the fibered product of two open immersions is their intersection.)

Hence the fibered product of a *quasiaffine* scheme (defined to be an open subscheme of an affine scheme) with an affine scheme over an affine scheme exists. *This isn't quite right; what we've shown, and what we'll use, is that the fibered product of a quasi-affine scheme with an affine scheme over an affine scheme Z exists so long as that quasi-affine scheme is an open subscheme of an affine scheme that also admits a map to Z extending the map from the quasiaffine. At some point I'll retype this to say this better. This sloppiness continues in later lectures, but the argument remains correct.*

Third, we show that $X \times_Z Y$ exists if Y and Z are affine and X is general. Before we show this, we remark that one special case of it is called "extension of scalars": if X is a k-scheme, and k' is a field extension (often k' is the algebraic closure of k), then $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ (sometimes informally written $X \times_k k'$ or $X_{k'}$) is a k'-scheme. Often properties of X can be checked by verifying them instead on $X_{k'}$. This is the subject of *descent* — certain properties "descend" from $X_{k'}$ to X.

Let's verify this. It will follow from abstract nonsense and the gluing lemma. Recall the *gluing lemma* (a homework problem): assume we are given a bunch of schemes X_i indexed by some index set I, along with open subschemes $U_{ij} \subset X_i$ indexed by $I \times I$, and isomorphisms $f_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$, satisfying the cocycle condition: $f_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $(f_{jk} \circ f_{ij})|_{U_{ij} \cap U_{ik}} = f_{ik}|_{U_{ij} \cap U_{ik}}$. Then they glue together to a unique scheme. (This was a homework problem long ago; I'll add a reference when I dig it up.)

We'll now apply this in our case. Cover X with affine open sets V_i. Let $V_{ij} = V_i \cap V_j$. Then for each of these, $X_i := V_i \times_Z Y$ exists, and each of them has open subsets $U_{ij} := V_{ij} \times_Z Y$, and isomorphisms satisfying the cocycle condition (because the V_i's and V_{ij}'s could be glued together via g_{ij} which satisfy the cocycle condition).

Call this glued-together scheme W. It comes with morphisms to X and Y (and their compositions to Z are the same). I claim that this satisfies the universal property for $X \times_Z Y$, basically because "morphisms glue" (yet another ancient exercise). Here's why. Suppose W' is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $W' \to W$ completing the diagram

$$\begin{tikzcd}
W' \arrow[dr, "\exists!?"] \arrow[drr, "g''"] \arrow[ddr, "f''"'] & & \\
& W \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z.
\end{tikzcd}$$

Now break W' up into open sets $W'_i = g''^{-1}(U_i)$. Then by the universal property for $V_i = U_i \times_Z Y$, there is a unique map $W'_i \to V_i$ (which we can interpret as $W'_i \to W$). (Thus we have already shown uniqueness of $W' \to W$.) These must agree on $W'_i \cap W'_j$, because there is only one map $W'_i \cap W'_j$ to W making the diagram commute (because of the second step — $(U_i \cap U_j) \times_Z Y$ exists). Thus all of these morphisms $W'_i \to W$ glue together; we have shown existence.

Fourth, we show that if Z is affine, and X and Y are arbitrary schemes, then $X \times_Z Y$ exists. We just repeat the process of the previous step, with the roles of X and Y repeated, using the fact that by the previous step, we can assume that the fibered product with an affine scheme with an arbitrary scheme over an affine scheme exists.

Fifth, we show that the fibered product of any two schemes over a *quasiaffine* scheme exists. Here is why: if $Z \hookrightarrow Z'$ is an open immersion into an affine scheme, then $X \times_Z Y = X \times_{Z'} Y$ are the same. (You can check this directly. But this is yet again an old exercise — problem set 1 problem A4 — following from the fact that $Z \hookrightarrow Z'$ is a monomorphism.)

Finally, we show that the fibered product of any scheme with any other scheme over any third scheme always exists. We do this in essentially the same way as the third step, using the gluing lemma and abstract nonsense. Say $f : X \to Z$, $g : Y \to Z$ are two morphisms of schemes. Cover Z with affine open subsets Z_i. Let $X_i = f^{-1}X_i$ and $Y_i = g^{-1}Y_i$. Define $Z_{ij} = Z_i \cap Z_j$, and X_{ij} and Y_{ij} analogously. Then $W_i := X_i \times_{Z_i} Y_i$ exists for all i, and has as open sets $W_{ij} := X_{ij} \times_{Z_{ij}} Y_{ij}$ along with gluing information satisfying the cocycle condition (arising from the gluing information for Z from the Z_i and Z_{ij}). Once again, we show that this satisfies the universal property. Suppose W' is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $W' \to W$ completing the diagram

$$\begin{tikzcd}
W' \arrow[dr, "\exists!?"] \arrow[drr, bend left, "g''"] \arrow[ddr, bend right, "f''"'] & & \\
& W \arrow[r, "f'"'] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z.
\end{tikzcd}$$

Now break W' up into open sets $W'_i = g'' \circ f^{-1}(Z_i)$. Then by the universal property for W_i, there is a unique map $W'_i \to W_i$ (which we can interpret as $W'_i \to W$). Thus we have already shown uniqueness of $W' \to W$. These must agree on $W'_i \cap W'_j$, because there is only one map $W'_i \cap W'_j$ to W making the diagram commute. Thus all of these morphisms $W'_i \to W$ glue together; we have shown existence. $\square$

3. Computing fibered products in practice

There are four types of morphisms that it is particularly easy to take fibered products with, and all morphisms can be built from these four atomic components.

(1) *base change by open immersions*

We've already done the work for this one, and we used it above.

$$\begin{tikzcd}
f^{-1}(Y) \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

I'll describe the remaining three on the level of affine sets, because we obtain general fibered products by gluing.

(2) *adding an extra variable*

Exercise. Show that $B \otimes_A A[t] \cong B[t]$.

Hence the following is a fibered diagram.

$$\begin{tikzcd}
\operatorname{Spec} B[t] \arrow[r] \arrow[d] & \operatorname{Spec} A[t] \arrow[d] \\
\operatorname{Spec} B \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

(3) *base change by closed immersions*

If the right column is obtained by modding out by a certain ideal (i.e. if the morphism is a closed immersion, i.e. if the map of rings in the other direction is surjective), then the left column is obtained by modding out by the pulled back elements of that ideal. In other words, if $T \to R, S$ are two ring morphisms, and I is an ideal of R, and I^e is the extension of I to $R \otimes_T S$ (the elements $\sum_j i_j \otimes s_j$, where $i_j \in I$ and $s_j \in S$, then there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

(This is precisely problem B3 on problem set 1.) Thus the natural morphism $R \otimes_T S \to R/I \otimes_T S$ is a surjection, and we have a base change diagram:

$$\begin{tikzcd}
\operatorname{Spec}(R \otimes_T S)/I^e \arrow[r] \arrow[d, hook] & \operatorname{Spec} R/I \arrow[d, hook] \\
\operatorname{Spec} R \otimes_T S \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\
\operatorname{Spec} S \arrow[r] & \operatorname{Spec} T
\end{tikzcd}$$

(where each rectangle is a fiber diagram).

Translation: the fibered product with a subscheme is the subscheme of the fibered product in the obvious way. We say that "closed immersions are preserved by base change".

(4) *base change by localization*

Exercise. Suppose $C \to B, A$ are two morphisms of rings. Suppose S is a multiplicative set of A. Then $(S \otimes 1)$ is a multiplicative set of $A \otimes_C B$. Show that there is a natural morphism $(S^{-1}A) \otimes_C B \cong (S \otimes 1)^{-1}(A \otimes_C B)$.

Hence we have a fiber diagram:

$$\begin{tikzcd}
\operatorname{Spec}(S \otimes 1)^{-1}(A \otimes_C B) \arrow[r] \arrow[d] & \operatorname{Spec} S^{-1}A \arrow[d] \\
\operatorname{Spec} A \otimes_C B \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec} B \arrow[r] & \operatorname{Spec} C
\end{tikzcd}$$

(where each rectangle is a fiber diagram).

Translation: the fibered product with a localization is the localization of the fibered product in the obvious way. We say that "localizations are preserved by base change". This is handy if the localization is of the form $A \hookrightarrow A_f$ (corresponding to taking distinguished open sets) or $A \hookrightarrow FF(A)$ (from A to the fraction field of A, corresponding to taking generic points), and various things in between.

These four tricks let you calculate lots of things in practice. For example,

$$\begin{aligned}&\operatorname{Spec} k[x_1,\dots,x_m]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m))\otimes_k\\ &\operatorname{Spec} k[y_1,\dots,y_n]/(g_1(y_1,\dots,y_n),\dots,g_s(y_1,\dots,y_n))\\ \cong\ &\operatorname{Spec} k[x_1,\dots,x_m,y_1,\dots,y_n]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m),\\ &g_1(y_1,\dots,y_n),\dots,g_s(y_1,\dots,y_n)).\end{aligned}$$

Here are many more examples.

4. Examples

One important example is of *fibers* of morphisms. Suppose $p \to Z$ is the inclusion of a point (not necessarily closed). Then if $g : Y \to Z$ is any morphism, the base change with $p \to Z$ is called the *fiber of* g *above* p or the *preimage of* p, and is denoted $g^{-1}(p)$. If Z is irreducible, the fiber above the generic point is called the *generic fiber*. In an affine open subscheme $\operatorname{Spec} A$ containing p, p corresponds to some prime ideal $\mathfrak{p}$, and the morphism corresponds to the ring map $A \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. this is the composition if localization and closed immersion, and thus can be computed by the tricks above.

Here is an interesting example, that we will consider multiple times during this course. Consider the projection of the parabola $y^2 = x$ to the x axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. (If $\mathbb{Q}$ alarms you, replace it with your favorite field and see what happens.)

Then the preimage of 1 is 2 points:

$$\begin{aligned}\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes_{\mathbb{Q}}\operatorname{Spec}\mathbb{Q}[x]/(x-1) &\cong \operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x-1)\\ &\cong \operatorname{Spec}\mathbb{Q}[y]/(y^2-1)\\ &\cong \operatorname{Spec}\mathbb{Q}[y]/(y-1)\coprod\operatorname{Spec}\mathbb{Q}[y]/(y+1).\end{aligned}$$

The preimage of 0 is 1 nonreduced point:

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x)\cong\operatorname{Spec}\mathbb{Q}[y]/(y^2).$$

The preimage of -1 is 1 reduced point, but of "size 2 over the base field".

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x+1)\cong\operatorname{Spec}\mathbb{Q}[y]/(y^2+1)\cong\operatorname{Spec}\mathbb{Q}[i].$$

The preimage of the generic fiber is again 1 reduced point, but of "size 2 over the residue field".

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes\mathbb{Q}(x)\cong\operatorname{Spec}\mathbb{Q}[y]\otimes\mathbb{Q}(y^2)$$

i.e. you take elements polynomials in y, and you are allowed to invert polynomials in y^2. A little thought shows you that you are then allowed to invert polynomials in y, as if $f(y)$ is any polynomial in y, then

$$\frac{1}{f(y)}=\frac{f(-y)}{f(y)f(-y)},$$

and the latter denominator is a polynomial in y^2. Thus

$$\operatorname{Spec} \mathbb{Q}[x, y]/(y^2 - x) \otimes \mathbb{Q}(x) \cong \mathbb{Q}(y)$$

which is a degree 2 field extension of $\mathbb{Q}(x)$.

For future reference notice the following interesting fact: in each case, the number of preimages can be interpreted as 2, where you count to two in several ways: you can count points; you can get non-reduced behavior; or you can have field extensions. This is going to be symptomatic of a very special and important kind of morphism (a finite flat morphism).

Here are some other examples.

4.1. *Exercise.* Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

4.2. *Exercise.* Show that for finite-type schemes over $\mathbb{C}$, the complex-valued points of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

Here is a definition in common use. The terminology is a bit unfortunate, because it is a second (different) meaning of "points of a scheme". If T is a scheme, the T-*valued points of a scheme* X are defined to be the morphism $T \to X$. They are sometimes denoted $X(T)$. If R is a ring (most commonly in this context a field), the R-*valued points of a scheme* X are defined to be the morphism $\operatorname{Spec} R \to X$. They are sometimes denoted $X(R)$. For example, if k is an algebraically closed field, then the k-valued points of a finite type scheme are just the closed points; but in general, things can be weirder. (When we say "points of a scheme", and not T-valued points, we will always mean the usual meaning, not this meaning.)

Exercise. Describe a natural bijection $(X \times_Z Y)(T) \cong X(T) \times_{Z(T)} Y(T)$. (The right side is a fibered product of sets.) In other words, fibered products behaves well with respect to T-valued points. This is one of the motivations for this notion.

4.3. *Exercise.* Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. This small example is the first case of something incredibly important.

4.4. *Exercise.* Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $t = u^2$. Show that $X \times_Y X$ has 2 irreducible components. Compare what is happening above the generic point of Y to the previous exercise.

4.5. *A little too vague to be an exercise.* More generally, suppose $K/\mathbb{Q}$ is a finite Galois field extension. Investigate the analogue of the previous two exercises. Try degree 2. Try degree 3.

4.6. *Hard but fascinating exercise for those familiar with the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$.* Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

4.7. *Exercise (A weird scheme).* Show that $\operatorname{Spec} \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ is an integral dimension one scheme, with closed points in natural correspondence with the transcendental complex numbers. (If the description $\operatorname{Spec} \mathbb{C}[t] \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$ is more striking, you can use that instead.) This scheme doesn't come up in nature, but it is certainly neat!

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