FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 20

CONTENTS

Last day: Maps to affine schemes; surjective, open immersion, closed immersion, quasicompact, locally of finite type, finite type, affine morphism, finite, quasifinite. Images of morphisms: constructible sets, and Chevalley's theorem (finite type morphism of Noetherian schemes sends constructibles to constructibles).

Today: Pushforwards and pullbacks of quasicoherent sheaves.

This is the last class of the first quarter of this three-quarter sequence. Last day, I defined a large number of classes of morphisms. Today, I will talk about how quasicoherent sheaves push forward or pullback. I'll then sum up what's happened in this class, and give you some idea of what will be coming in the next quarter.

1. PUSHFORWARDS AND PULLBACKS OF QUASICOHERENT SHEAVES

There are two things you can do with modules and a ring homomorphism $B \to A$. If M is an A-module, you can create an B-module M_B by simply treating it as an B-module. If N is an B-module, you can create an A-module $N \otimes_B A$.

These notions behave well with respect to localization (in a way that we will soon make precise), and hence work (often) in the category of quasicoherent sheaves. The two functors are adjoint:

$$\mathrm{Hom}_A(N \otimes_B A, M) \cong \mathrm{Hom}_B(N, M_B)$$

(where this isomorphism of groups is functorial in both arguments), and we will see that this remains true on the scheme level.

One of these constructions will turn into our old friend pushforward. The other will be a relative of pullback, whom I'm reluctant to call an "old friend".

Date: Friday, December 9, 2005. Last small update June 25, 2007.

2. Pushforwards of quasicoherent sheaves

The main message of this section is that in "reasonable" situations, the pushforward of a quasicoherent sheaf is quasicoherent, and that this can be understood in terms of one of the module constructions defined above. We begin with a motivating example:

2.1. *Exercise.* Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and suppose M is an A-module, so $\tilde{M}$ is a quasicoherent sheaf on $\operatorname{Spec} A$. Show that $f_* \tilde{M} \cong \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished opens!)

In particular, $f_*\tilde{M}$ is quasicoherent. Perhaps more important, this implies that the pushforward of a quasicoherent sheaf under an affine morphism is also quasicoherent. The following result doesn't quite generalize this statement, but the argument does.

2.2. *Theorem. — Suppose $f : X \to Y$ is a morphism, and X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Then $f_*\mathcal{F}$ is a quasicoherent sheaf on Y.*

The fact about f that we will use is that the preimage of any affine open subset of Y is a finite union of affine sets (f is quasicompact), and the intersection of any two of these affine sets is also a finite union of affine sets (this is a definition of the notion of a *quasiseparated morphism*). Thus the "correct" hypothesis here is that f is quasicompact and quasiseparated.

Proof. By the first definition of quasicoherent sheaves, it suffices to show the following: if $\mathcal{F}$ is a quasicoherent sheaf on X, and $f : X \to \operatorname{Spec} R$, then the following diagram commutes:

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{\operatorname{res}_{D(g) \subset \operatorname{Spec} R}} & \Gamma(X_g, \mathcal{F}) \\ & {\scriptstyle \otimes_R R_g} \searrow \quad \nearrow {\scriptstyle \sim} & \\ & \Gamma(X, \mathcal{F})_g & \end{array}$$

This was a homework problem (# 18 on problem set 6)! ☐

2.3. *Exercise.* Give an example of a morphism of schemes $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X such that $\pi_*\mathcal{F}$ is not quasicoherent. (Answer: $Y = \mathbb{A}^1$, $X =$ countably many copies of $\mathbb{A}^1$. Then let $f = t$. X_t has a global section $(1/t, 1/t^2, 1/t^3, \dots)$. The key point here is that infinite direct sums do not commute with localization.)

Coherent sheaves don't always push forward to coherent sheaves. For example, consider the structure morphism $f : \mathbb{A}^1_k \to \operatorname{Spec} k$, given by $k \mapsto k[t]$. Then $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is the $k[t]$, which is not a finitely generated k-module. Under especially good situations, coherent sheaves do push forward. For example:

2.4. *Exercise.* Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_*\mathcal{F}$ is a coherent sheaf. (Hint: Show first that $f_*\mathcal{O}_X$ is finite type = locally finitely generated.)

Once we define cohomology of quasicoherent sheaves, we will quickly prove that if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_k$, then $\Gamma(\mathbb{P}^n_k)$ is a finite-dimensional k-module, and more generally if $\mathcal{F}$ is a coherent sheaf on $\operatorname{Proj} S_*$, then $\Gamma(\operatorname{Proj} S_*)$ is a coherent A-module (where $S_0 = A$). This is a special case of the fact the "pushforwards of coherent sheaves by projective morphisms are also coherent sheaves". We will first need to define "projective morphism"! This notion is a generalization of $\operatorname{Proj} S_* \to \operatorname{Spec} A$.

3. Pullback of quasicoherent sheaves

(Note added in February: I will try to reserve the phrase "pullback of a sheaf" for pullbacks of quasicoherent sheaves f^*, and "inverse image sheaf" for f^{-1}, which applies in a more general situation, in the category of sheaves on topological spaces.)

I will give four definitions of the pullback of a quasicoherent sheaf. The first one is the most useful in practice, and is in keeping with our emphasis of quasicoherent sheaves as just "modules glued together". The second is the "correct" definition, as an adjoint of pushforward. The third, which we mention only briefly, is *more* correct, as adjoint in the category of $\mathcal{O}_X$-modules. And we end with a fourth definition.

We note here that pullback to a closed subscheme or an open subscheme is often called **restriction**.

3.1. Construction/description of the pullback. Let us now define the pullback functor precisely. Suppose $X \to Y$ is a morphism of schemes, and $\mathcal{G}$ is a quasicoherent sheaf on Y. We will describe the pullback quasicoherent sheaf $f^*\mathcal{G}$ on X by describing it as a sheaf on the distinguished affine base. In our base, we will permit only those affine open sets $U \subset X$ such that $f(U)$ is contained in an affine open set of Y. The distinguished restriction map will force this sheaf to be quasicoherent.

Suppose $U \subset X$, $V \subset Y$ are affine open sets, with $f(U) \subset V$, $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$. Suppose $\mathcal{F}|_V \cong \tilde{N}$. Then define $\Gamma(f^*_V\mathcal{F}, U) := N \otimes_B A$. Our main goal will be to show that this is independent of our choice of V.

We begin as follows: we fix an affine open subset $V \subset Y$, and use it to define sections over any affine open subset $U \subset f^{-1}(V)$. We show that this gives us a quasicoherent sheaf $f^*_V\mathcal{G}$ on $f^{-1}(V)$, by showing that these sections behave well with respect to distinguished restrictions. First, note that if $D(f) \subset U$ is a distinguished open set, then

$$\Gamma(f^*_V\mathcal{F}, D(f)) = N \otimes_B A_f \cong (N \otimes_B A) \otimes_A A_f = \Gamma(f^*_V\mathcal{F}, U) \otimes_A A_f.$$

Define the restriction map $\Gamma(f^*_V\mathcal{F}, U) \to \Gamma(f^*_V\mathcal{F}, D(f))$ by

$$\Gamma(f^*_V\mathcal{F}, U) \to \Gamma(f^*_V\mathcal{F}, U) \otimes_A A_f \tag{1}$$

(with $\alpha \mapsto \alpha \otimes 1$ of course). Thus on the *distinguished affine topology of* $\operatorname{Spec} A$ we have defined a quasicoherent sheaf.

Finally, we show that if $f(U)$ is contained in *two* affine open sets V_1 and V_2, then the alleged sections of the pullback we have described do not depend on whether we use V_1 or V_2. More precisely, we wish to show that

$$\Gamma(f^*_{V_1}\mathcal{F}, U) \quad \text{and} \quad \Gamma(f^*_{V_2}\mathcal{F}, U)$$

have a canonical isomorphism, which commutes with the restriction map (1).

Let $\{W_i\}_{i \in I}$ be an affine cover of $V_1 \cap V_2$ by sets that are distinguished in *both* V_1 and V_2 (possible by the Proposition we used in the proof of the Affine Communication Lemma). Then by the previous paragraph, as $f^*_{V_1}\mathcal{F}$ is a sheaf on the distinguished base of V_1,

$$\Gamma(f^*_{V_1}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i \cap W_j))\right).$$

If $V_1 = \operatorname{Spec} B_1$ and $W_i = D(g_i)$, then

$$\Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i)) = N \otimes_{B_1} A_{f^\# g_i} \cong N \otimes_{(B_1)_{g_i}} A_{f^\# g_i} = \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)),$$

so

$$\Gamma(f^*_{V_1}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i \cap W_j))\right). \tag{2}$$

The same argument for V_2 yields

$$\Gamma(f^*_{V_2}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i \cap W_j))\right). \tag{3}$$

But the right sides of (2) and (3) are the same, so the left sides are too. Moreover, (2) and (3) behave well with respect to restricting to a distinguished open $D(g)$ of $\operatorname{Spec} A$ (just apply $\otimes_A A_g$ to the the right side) so we are done.

Hence we have described a quasicoherent sheaf $f^*\mathcal{G}$ on X whose behavior on affines mapping to affines was as promised.

3.2. *Theorem. —*

1. *The pullback of the structure sheaf is the structure sheaf.*
2. *The pullback of a finite type (=locally finitely generated) sheaf is finite type.*
3. *The pullback of a finitely presented sheaf is finitely presented. Hence if $f : X \to Y$ is a morphism of locally Noetherian schemes, then the pullback of a coherent sheaf is coherent. (It is not always true that the pullback of a coherent sheaf is coherent, and the interested reader can think of a counterexample.)*
4. *The pullback of a locally free sheaf of rank r is another such. (In particular, the pullback of an invertible sheaf is invertible.)*
5. *(functoriality in the morphism) $\pi_1^*\pi_2^*\mathcal{F} \cong (\pi_2 \circ \pi_1)^*\mathcal{F}$*
6. *(functoriality in the quasicoherent sheaf) $\mathcal{F}_1 \to \mathcal{F}_2$ induces $\pi^*\mathcal{F}_1 \to \pi^*\mathcal{F}_2$*
7. *If s is a section of $\mathcal{F}$ then there is a natural section π^*s that is a section of $\pi^*\mathcal{F}$.*
8. *(stalks) If $\pi : X \to Y$, $\pi(x) = y$, then $(\pi^*\mathcal{F})_x \cong \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$. The previous map, restricted to the stalks, is $f \mapsto f \otimes 1$. (In particular, the locus where the section on the target vanishes pulls back to the locus on the source where the pulled back section vanishes.)*

(9) *(fibers) Pullbacks of fibers are given as follows: if* $\pi : X \to Y$, *where* $\pi(x) = y$, *then* $\pi^*\mathcal{F}/\mathfrak{m}_{X,x}\pi^*\mathcal{F} \cong (\mathcal{F}/\mathfrak{m}_{Y,y}\mathcal{F}) \otimes_{\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$
(10) *(tensor product)* $\pi^*(\mathcal{F} \otimes \mathcal{G}) = \pi^*\mathcal{F} \otimes \pi^*\mathcal{G}$
(11) *pullback is a right-exact functor*

All of the above are interconnected in obvious ways. For example, given $\mathcal{F}_1 \to \mathcal{F}_2$ and a section s of $\mathcal{F}_1$, then we can pull back the section and then send it to $\pi^*\mathcal{F}_2$, or vice versa, and we get the same thing.

I used some of these results to help give an intuitive picture of the pullback.

Proof. Most of these are left to the reader. It is convenient to do right-exactness early (e.g. before showing that finitely presented sheaves pull back to finitely presented sheaves). For the tensor product fact, show that $(M \otimes_S R) \otimes (N \otimes_S R) \cong (M \otimes N) \otimes_S R$, and that this behaves well with respect to localization. The proof of the fiber fact is as follows. $(S, \mathfrak{n}) \to (R, \mathfrak{m})$.

$$\begin{array}{ccc} S & \longrightarrow & R \\ \downarrow & & \downarrow \\ S/\mathfrak{n} & \longrightarrow & R/\mathfrak{m} \end{array}$$

$(N \otimes_S R) \otimes_R (R/\mathfrak{m}) \cong (N \otimes_S (S/\mathfrak{n})) \otimes_{S/\mathfrak{n}} (R/\mathfrak{m})$ as both sides are isomorphic to $N \otimes_S (R/\mathfrak{m})$. □

3.3. *Unimportant Exercise.* Verify that the following is an example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where p is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_p \to 0.$$

(This is a closed subscheme exact sequence; also an effective Cartier exact sequence. Algebraically, we have $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$.) Restrict to p.

3.4. Pulling back closed subschemes. Suppose $Z \hookrightarrow Y$ is a closed immersion, and $X \to Y$ is any morphism. Then we define the pullback of the closed subscheme Z to X as follows. We pullback the quasicoherent sheaf of ideals on Y defining Z to get a quasicoherent sheaf of ideals on X (which we take to define W). Equivalently, on any affine open Y, Z is cut out by some functions; we pull back those functions to X, and denote the scheme cut out by them by W.

Exercise. Let W be the pullback of the closed subscheme Z to X. Show that $W \cong Z \times_Y X$. In other words, the fibered product with a closed immersion always exists, and closed immersions are preserved by fibered product (or by pullback), i.e. if

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ Z & \xrightarrow{g} & Y \end{array}$$

is a fiber diagram, and g is a closed immersion, then so is g'. (This is actually a repeat of an exercise in class 19 — sorry!)

3.5. Three more "definitions". Pullback is left-adjoint of the pushforward. This is a theorem (which we'll soon prove), but it is actually a pretty good definition. If it exists, then it is unique up to unique isomorphism by Yoneda nonsense.

The problem is this: pushforwards don't always exist (in the category of quasicoherent sheaves); we need the quasicompact and quasiseparated hypotheses. However, pullbacks always exist. So we need to motivate our definition of pullback even without the quasicompact and quasiseparated hypothesis. (One possible motivation will be given in Remark 3.7.)

3.6. *Theorem. — Suppose* $\pi : X \to Y$ *is a quasicompact, quasiseparated morphism. Then pullback is left-adjoint to pushforward. More precisely,* $\mathrm{Hom}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$.

(The quasicompact and quasiseparated hypothesis is required to ensure that the pushforward exists, not because it is needed in the proof.)

More precisely still, we describe natural homomorphisms that are functorial in both arguments. We show that it is a bijection of sets, but it is fairly straightforward to verify that it is an isomorphism of groups. Not surprisingly, we will use adjointness for modules.

Proof. Let's unpack the right side. What's an element of $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$? For every affine V in Y, we get an element of $\mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(f^{-1}(V)))$, and this behaves well with respect to distinguished opens. Equivalently, for every affine V in Y and U in $f^{-1}(V) \subset X$, we have an element $\mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(U))$, that behaves well with respect to localization to distinguished opens on both affines. By the adjoint property, this corresponds to elements of $\mathrm{Hom}(\mathcal{G}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U), \mathcal{F}(U))$, which behave well with respect to localization. And that's the left side. $\square$

3.7. *Pullback for ringed spaces.* (This is actually conceptually important but distracting for our exposition; we encourage the reader to skip this, at least on the first reading.) Pullbacks and pushforwards may be defined in the category of modules over ringed spaces. We define pushforward in the usual way, and then define the pullback of an $\mathcal{O}_Y$-module using the adjoint property. Then one must show that (i) it exists, and (ii) the pullback of a quasicoherent sheaf is quasicoherent. The fourth definition is as follows: suppose we have a morphism of ringed spaces $\pi : X \to Y$, and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then we define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. We will not show that this definition is equivalent to ours, but the interested reader is welcome to try this as an exercise. There is probably a proof in Hartshorne. The statements of Theorem 3.6 apply in this more general setting. (Really the third definition "requires" the fourth.)

Here is a hint as to why this definition is equivalent to ours (a hint for the exercise if you will). We need to show that $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ ("definition 4") and $f^*\mathcal{F}$ ("definition 1") are isomorphic. You should (1) find a natural morphism from one to the other, and (2) show that it is an isomorphism at the level of stalks. The difficulty of (1) shows the disadvantages of our definition of quasicoherent sheaves.

E-mail address: `vakil@math.stanford.edu`