FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 18

CONTENTS

Last day: Associated points; more on normality; invertible sheaves and divisors take 1.

Today: Invertible sheaves and divisors. Morphisms of schemes.

1. INVERTIBLE SHEAVES AND DIVISORS

We next develop some mechanism of understanding invertible sheaves (line bundles) on a given scheme X. Define $\operatorname{Pic} X$ to be the group of invertible sheaves on X. How can we describe many of them? How can we describe them all? Our goal for the first part of today will be to partially address this question. As an important example, we'll show that we have already found all the invertible sheaves on projective space $\mathbb{P}^n_k$ — they are the $\mathcal{O}(m)$.

One moral of this story will be that invertible sheaves will correspond to "codimension 1 information".

Recall one way of getting invertible sheaves, by way of *effective Cartier divisors*. Recall that an effective Cartier divisor is a closed subscheme such that there exists an affine cover such that on each one it is cut out by a single equation, not a zero-divisor. (This does not mean that on *any* affine it is cut out by a single equation — this notion doesn't satisfy the "gluability" hypothesis of the Affine Communication Lemma. If $I \subset R$ is generated by a non-zero divisor, then $I_f \subset R_f$ is too. But "not conversely". I might give an example later, involving an elliptic curve.) By Krull's Principal Ideal Theorem, it is pure codimension 1.

Remark: if $I = (u) = (v)$, and u is not a zero-divisor, then u and v differ multiplicatively by a unit in R. Proof: $u \in (v)$ implies $u = av$. Similarly $v = bu$. Thus $u = abu$, from which $u(1 - ab) = 0$. As u is not a zero-divisor, $1 = ab$, so a and b are units. In other words, the generator of such an ideal is well-defined up to a unit.

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The reason we care: effective Cartier divisors give invertible sheaves. If $\mathcal{I}$ is an effective Cartier divisor on X, then $\mathcal{I}$ is an invertible sheaf. Reason: locally, sections are multiples of a single generator u, and there are no "relations".

Recall that the invertible sheaf $\mathcal{O}(D)$ corresponding to an effective Cartier divisor is defined to be the *dual* of the ideal sheaf $\mathcal{I}_D$. The ideal sheaf itself is sometimes denoted $\mathcal{O}(-D)$. We have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

The invertible sheaf $\mathcal{O}(D)$ has a canonical section: Dualizing $0 \to \mathcal{I} \to \mathcal{O}$ gives us $\mathcal{O} \to \mathcal{I}^*$.

Exercise. This section vanishes along our actual effective Cartier divisor.

Exercise. Conversely, if $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zero divisor (make sense of this!), then $s = 0$ cuts out an effective Cartier divisor D, and $\mathcal{O}(D) \cong \mathcal{L}$. (If X is locally Noetherian, "not locally a zero divisor" translate to "does not vanish at an associated point".)

Define the *sum* of two effective Cartier divisors as follows: if $I = (u)$ (locally) and $J = (v)$, then the sum corresponds to (uv) locally. (Verify that this is well-defined!)

Exercise. Show that $\mathcal{O}(D + E) \cong \mathcal{O}(D) \otimes \mathcal{O}(E)$.

Thus we have a map of semigroups, from effective Cartier divisors to invertible sheaves with sections not locally zero-divisors (and hence also to the Picard group of invertible sheaves).

Hence we can get a bunch of invertible sheaves, by taking differences of these two. The surprising fact: we "usually get them all"! In fact it is very hard to describe an invertible sheaf on a finite type k-scheme that is not describable in such a way (we will see later today that there are none if the scheme is nonsingular or even factorial; and we might see later in the year that there are none if the scheme is quasiprojective).

Instead, I want to take another tack. Some of what we do will generalize to the non-normal case, which is certainly important, and experts are invited to think about this.

Define a *Weil divisor* as a formal sum of height 1 irreducible closed subsets of X. (This makes sense more generally on any pure dimensional, or even locally equidimensional, scheme.) In other words, a Weil divisor is defined to be an object of the form

$$\sum_{Y \subset X \text{ height } 1} n_Y [Y]$$

the n_Y are integers, all but a finite number of which are zero. Weil divisors obviously form an abelian group, denoted $\operatorname{Weil} X$.

A Weil divisor is said to be *effective* if $n_Y \geq 0$ for all Y. In this case we say $D \geq 0$, and by $D_1 \geq D_2$ we mean $D_1 - D_2 \geq 0$. The *support* of a Weil divisor D is the subset

$\cup_{n_Y \neq 0} Y$. If $U \subset X$ is an open set, there is a natural restriction map $\operatorname{Weil} X \to \operatorname{Weil} U$, where $\sum n_Y [Y] \mapsto \sum_{Y \cap U \neq \emptyset} n_Y [Y \cap U]$.

Suppose now that X is a Noetherian scheme, regular in codimension 1. We add this hypothesis because we will use properties of discrete valuation rings. Suppose that $\mathcal{L}$ is an invertible sheaf, and s a rational section not vanishing on any irreducible component of X. Then s determines a Weil divisor

$$\operatorname{div}(s) := \sum_Y \operatorname{val}_Y(s)[Y].$$

(Recall that $\operatorname{val}_Y(s) = 0$ for all but finitely many Y, by problem 46 on problem set 5.) This is the "divisor of poles and zeros of s". (To determine the valuation $\operatorname{val}_Y(s)$ of s along Y, take any open set U containing the generic point of Y where $\mathcal{L}$ is trivializable, along with any trivialization over U; under this trivialization, s is a function on U, which thus has a valuation. Any two such trivializations differ by a unit, so this valuation is well-defined.) This map gives a group homomorphism

(1)

$$\operatorname{div} : \{(\text{invertible sheaf } \mathcal{L}, \text{rational section } s \text{ not vanishing at any minimal prime})\}/\Gamma(X, \mathcal{O}_X^*) \to \operatorname{Weil} X.$$

1.1. *Exercise.* (a) (divisors of rational functions) Verify that on $\mathbb{A}^1_k$, $\operatorname{div}(x^3/(x+1)) = 3[(x)] - [(x+1)] = 3[0] - [-1]$.
(b) (divisor of a rational sections of a nontrivial invertible sheaf) Verify that on $\mathbb{P}^1_k$, there is a rational section of $\mathcal{O}(1)$ "corresponding to" x^2/y. Calculate $\operatorname{div}(x^2/y)$.

We want to classify all invertible sheaves on X, and this homomorphism (1) will be the key. Note that any invertible sheaf will have such a rational section (for each irreducible component, take a non-empty open set not meeting any other irreducible component; then shrink it so that $\mathcal{L}$ is trivial; choose a trivialization; then take the union of all these open sets, and choose the section on this union corresponding to 1 under the trivialization). We will see that in reasonable situations, this map div will be injective, and often even an isomorphism. Thus by forgetting the rational section (taking an appropriate quotient), we will have described the Picard group. Let's put this strategy into action. *Suppose from now on that X is normal.*

1.2. *Proposition. — The map* div *is injective.*

Proof. Suppose $\operatorname{div}(\mathcal{L}, s) = 0$. Then s has no poles. Hence by Hartogs' theorem, s is a regular section. Now s vanishes nowhere, so s gives an isomorphism $\mathcal{O}_X \to \mathcal{L}$ (given by $1 \mapsto s$). □

Motivated by this, we try to find the inverse map to div.

Definition. Suppose D is a Weil divisor. If $U \subset X$ is an open subscheme, define $\operatorname{Frac}(U)$ to be the field of total fractions of U, i.e. the product of the stalks at the minimal primes of U. (As described earlier, if U is irreducible, this is the function field.) Define $\operatorname{Frac}(U)^*$ to be those rational functions not vanishing at any generic point of U (i.e. not vanishing on any irreducible component of U). Define the sheaf $\mathcal{O}(D)$ by

$$\Gamma(U, \mathcal{O}(D)) := \{s \in \mathrm{Frac}(U)^* : \operatorname{div} s + D|_U \geq 0\}.$$

Note that the sheaf $\mathcal{O}(D)$ has a canonical rational section, corresponding to $1 \in \mathrm{Frac}(U)^*$.

1.3. *Proposition. — Suppose $\mathcal{L}$ is an invertible sheaf, and s is a rational section not vanishing on any irreducible component of X. Then there is an isomorphism $(\mathcal{L}, s) \cong (\mathcal{O}(\operatorname{div} s), t)$, where t is the canonical section described above.*

Proof. We first describe the isomorphism $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$. Over open subscheme $U \subset X$, we have a bijection $\Gamma(U, \mathcal{L}) \to \Gamma(U, \mathcal{O}(\operatorname{div} s))$ given by $s' \mapsto s'/s$, with inverse obviously given by $t' \mapsto st'$. Clearly under this bijection, s corresponds to the section 1 in $\mathrm{Frac}(U)^*$; this is the section we are calling t. □

We denote the subgroup of $\operatorname{Weil} X$ corresponding to divisors of rational *functions* the subgroup of *principal divisors*, which we denote $\operatorname{Prin} X$. Define the *class group* of X, $\operatorname{Cl} X$, by $\operatorname{Weil} X / \operatorname{Prin} X$. By taking the quotient of the inclusion (1) by $\operatorname{Prin} X$, we have the inclusion

$$\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X.$$

We're now ready to get a hold of $\operatorname{Pic} X$ rather explicitly!

First, some algebraic preliminaries.

1.4. *Exercise.* Suppose that A is a Noetherian domain. Show that A is a Unique Factorization Domain if and only if all height 1 primes are principal. You can use this to answer that homework problem, about showing that $k[w, x, y, z]/(wz - xy)$ is not a Unique Factorization Domain.

1.5. *Exercise.* Suppose that A is a Noetherian domain. Show that A is a Unique Factorization Domain if and only if A is integrally closed and $\operatorname{Cl} \operatorname{Spec} A = 0$. (One direction is easy: we have already shown that Unique Factorization Domains are integrally closed in their fraction fields. Also, the previous exercise shows that all height 1 primes are principal, so that implies that $\operatorname{Cl} \operatorname{Spec} A = 0$. It remains to show that if A is integrally closed and $\operatorname{Cl} X = 0$, then all height 1 prime ideals are principal. "Hartogs" may arise in your argument.)

Hence $\operatorname{Cl}(\mathbb{A}^n_k) = 0$, so $\boxed{\operatorname{Pic}(\mathbb{A}^n_k) = 0}$. (Geometers will find this believable: "$\mathbb{C}^n$ is a contractible manifold, and hence should have no nontrivial line bundles".)

Another handy trick is the following. Suppose Z is an irreducible codimension 1 subset of X. Then we clearly have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Weil} X \longrightarrow \operatorname{Weil}(X - Z) \longrightarrow 0.$$

When we take the quotient by principal divisors, we get:

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl} X \longrightarrow \operatorname{Cl}(X - Z) \longrightarrow 0.$$

For example, let $X = \mathbb{P}^n_k$, and Z be the hyperplane $x_0 = 0$. We have

$$\mathbb{Z} \to \operatorname{Cl} \mathbb{P}^n_k \to \operatorname{Cl} \mathbb{A}^n_k \to 0$$

from which $\operatorname{Cl} \mathbb{P}^n_k = \mathbb{Z}[Z]$ (which is $\mathbb{Z}$ or 0), and $\operatorname{Pic} \mathbb{P}^n_k$ is a subgroup of this.

1.6. *Important exercise.* Verify that $[Z] \to \mathcal{O}(1)$. In other words, let Z be the Cartier divisor $x_0 = 0$. Show that $\mathcal{O}(Z) \cong \mathcal{O}(1)$. (For this reason, people sometimes call $\mathcal{O}(1)$ the *hyperplane class* in $\operatorname{Pic} X$.)

Hence $\operatorname{Pic} \mathbb{P}^n_k \hookrightarrow \operatorname{Cl} \mathbb{P}^n_k$ is an isomorphism, and $\boxed{\operatorname{Pic} \mathbb{P}^n_k \cong \mathbb{Z}}$, with generator $\mathcal{O}(1)$. The *degree* of an invertible sheaf on $\mathbb{P}^n$ is defined using this: the degree of $\mathcal{O}(d)$ is of course d.

More generally, if X is *factorial* — all stalks are Unique Factorization Domains — then for any Weil divisor D, $\mathcal{O}(D)$ is invertible, and hence the map $\operatorname{Pic} X \to \operatorname{Cl} X$ is an isomorphism. (Proof: It will suffice to show that $[Y]$ is Cartier if Y is any irreducible codimension 1 set. Our goal is to cover X by open sets so that on each open set U there is a function whose divisor is $[Y \cap U]$. One open set will be $X - Y$, where we take the function 1. Next, suppose $x \in Y$; we will find an open set $U \subset X$ containing x, and a function on it. As $\mathcal{O}_{X,x}$ is a unique factorization domain, the prime corresponding to 1 is height 1 and hence principal (by Exercise 1.4). Let $f \in \operatorname{Frac} A$ be a generator. Then f is regular at x. f has a finite number of zeros and poles, and through x there is only one 0, notably $[Y]$. Let U be X minus all the others zeros and poles.)

I will now mention a bunch of other examples of class groups and Picard groups you can calculate.

For the first, I want to note that you can restrict invertible sheaves on X to any subscheme Y, and this can be a handy way of checking that an invertible sheaf is not trivial. For example, if X is something crazy, and $Y \cong \mathbb{P}^1$, then we're happy, because we understand invertible sheaves on $\mathbb{P}^1$. Effective Cartier divisors sometimes restrict too: if you have effective Cartier divisor on X, then it restricts to a closed subscheme on Y, locally cut out by one equation. If you are fortunate that this equation doesn't vanish on any associated point of Y, then you get an effective Cartier divisor on Y. You can check that the restriction of effective Cartier divisors corresponds to restriction of invertible sheaves.

1.7. *Exercise: a torsion Picard group.* Show that Y is an irreducible degree d hypersurface of $\mathbb{P}^n$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$. (For differential geometers: this is related to the fact that $\pi_1(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$.)

1.8. *Exercise.* Let $X = \operatorname{Proj} k[w, x, y, z]/(wz - xy)$, a smooth quadric surface. Show that $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows: Show that if L and M are two lines in different rulings (e.g. $L = V(w, x)$ and $M = V(w, y)$), then $X - L - M \cong \mathbb{A}^2$. This will give you a surjection

$\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \mathrm{Cl}\, X$. Show that $\mathcal{O}(L)$ restricts to $\mathcal{O}$ on L and $\mathcal{O}(1)$ on M. Show that $\mathcal{O}(M)$ restricts to $\mathcal{O}$ on M and $\mathcal{O}(1)$ on L. (This is a bit longer to do, but enlightening.)

1.9. *Exercise.* Let $X = \mathrm{Spec}\, k[w, x, y, z]/(xy - z^2)$, a cone. show that $\mathrm{Pic}\, X = 1$, and $\mathrm{Cl}\, X \cong \mathbb{Z}/2$. (Hint: show that the ruling $Z = \{x = z = 0\}$ generates $\mathrm{Cl}\, X$ by showing that its complement is isomorphic to $\mathbb{A}^2_k$. Show that $2[Z] = \mathrm{div}(x)$ (and hence principal), and that Z is not principal (an example we did when discovering the power of the Zariski tangent space).

Note: on curves, the invertible sheaves correspond to formal sums of points, modulo equivalence relation.

Number theorists should note that we have recovered a common description of the class group: formal sums of primes, modulo an equivalence relation.

Remark: Much of this discussion works without the hypothesis of normality, and indeed because non-normal schemes come up all the time, we need this additional generality. Think through this if you like.

2. Morphisms of schemes

Here are two motivations that will "glue together".

(a) We'll want morphisms of affine schemes $\mathrm{Spec}\, R \to \mathrm{Spec}\, S$ to be precisely the ring maps $S \to R$. Then we'll want maps of schemes to be things that "look like this". "the category of affine schemes is opposite to the category of rings". More correctly there is an equivalence of categories...

(b) We are also motivated by the theory of differentiable manifolds. We'll want a continuous maps from the underlying topological spaces $f : X \to Y$, along with a "pullback morphism" $f^\# : \mathcal{O}_S \to f_* \mathcal{O}_X$. There are many things we'll want to be true, that seem make a tall order; a clever idea will give us all of this for free. (i) Certainly values at points should map. They can't be the same: $\mathrm{Spec}\, \mathbb{C} \to \mathrm{Spec}\, \mathbb{R}$. (ii) $\mathrm{Spec}\, k[\epsilon]/\epsilon^2 \to \mathrm{Spec}\, k[\delta]/\delta^2$ is given by a map $\delta \mapsto q\epsilon$. These aren't distinguished by maps on points. (iii) Suppose you have a function σ on Y (i.e. $\sigma \in \Gamma(Y, \mathcal{O}_Y)$. Then it will pull back to a function $f^{-1}(\sigma)$ on X. However we make sense of pullbacks of functions (i) and (ii), certainly the locus where $f^{-1}(\sigma)$ vanishes on X should be the pullback of the locus where σ vanishes on Y. This will imply that the maps on stalks will be a local map (if $f(p) = q$ then $f^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ sends the maximal ideal. translating to: then germs of functions vanishing at q pullback to germs of functions vanishing at p). This last thing does it for us.

3. Ringed spaces and their morphisms

A ringed space is a topological space X along with a sheaf $\mathcal{O}_X$ of rings (called the *structure sheaf*. Our central example is a scheme. Another example is a differentiable manifold with the analytic topology and the sheaf of differentiable functions.

A **morphism of ringed spaces** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ (also sloppily denoted by the same name "f") along with a morphism $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves on Y (or equivalently but less usefully $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X, by adjointness). The morphism is often denoted $X \to Y$ when the structure sheaves and morphisms "between" them are clear from the context. There is an obvious notion of composition of morphisms; hence there is a category of ringed spaces. Hence we have notion of automorphisms and isomorphisms.

Slightly unfortunate notation: $f : X \to Y$ often denotes everything. Also used for maps of underlying sets, or underlying topological spaces. Usually clear from context.

3.1. *Exercise.* If $W \subset X$ and $Y \subset Z$ are both open immersions of ringed spaces, show that any morphism of ringed spaces $X \to Y$ induces a morphism of ringed spaces $W \to Z$.

3.2. *Exercise.* Show that morphisms of ringed spaces glue. In other words, suppose X and Y are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (Long ago we had an exercise proving this for topological spaces.)

3.3. *Easy important exercise.* Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

3.4. Important Example. Suppose $f^\# : S \to R$ is a morphism of rings. Define a morphism of ringed spaces as follows. $f : sp(\operatorname{Spec} R) \to sp(\operatorname{Spec} S)$. First as sets. $\mathfrak{p}$ prime in S, then ${f^\#}^{-1}(\mathfrak{p})$ is prime in R.

We interrupt this definition for a picture: $R = \operatorname{Spec} k[x, y]$, $S = \operatorname{Spec} k[t]$, $t \mapsto x$. Draw picture. Look at primes $(x - 2, y - 3)$. Look at (0). Look at $(x - 3)$. $(y - x^2)$.

It's a continuous map of topological spaces: $D(s)$ pulls back to $D(f^\# s)$. Now the map on sheaves. If $s \in S$, then show that $\Gamma(D(s), f_* \mathcal{O}_R) = R_{f^\# s} \cong R \otimes_S S_s$. (**Exercise.** Verify that $R_{f^\# s} \cong R \otimes_S S_s$ if you haven't seen this before.) Show that $f_* : \Gamma(D(s), \mathcal{O}_S) = S_s \to \Gamma(D(s), f_* \mathcal{O}_R) = R \otimes_S S_s$ given by $s' \mapsto 1 \otimes s'$ is a morphism of sheaves on the distinguished base of S, and hence defines a morphism of sheaves $f_* \mathcal{O}_R \to \mathcal{O}_S$.

4. Definition of morphisms of schemes

A morphism $f : X \to Y$ of schemes is a morphism of ringed spaces. Sadly, if X and Y are schemes, then there are morphisms $X \to Y$ *as ringed spaces* that are not morphisms as schemes. (See Example II.2.3.2 in Hartshorne for an example.)

The idea behind definition of morphisms is as follows. We define morphisms of affine schemes as in Important Example 3.4. (Note that the category of affine schemes is "opposite to the category of rings": given a morphisms of schemes, we get a map of rings in the opposite direction, and vice versa.)

4.1. *Definition/Proposition.* — A morphism of schemes $f : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affines. In other words, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces (Exercise 3.1) is a morphism of affine schemes. It suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover X.

We could prove the proposition using the affine communication theorem, but there's a clever trick. For this we need a digression on locally ringed spaces. They will not be used hereafter.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,x}$ are all local rings. A *morphism of locally ringed spaces* $f : X \to Y$ is a morphism of ringed spaces such that the induced map of stalks (Exercise 3.3) $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ sends the maximal ideal of the former to the maximal ideal of the latter. (This is sometimes called a "local morphism of local rings".) This means something rather concrete and intuitive: "if $p \mapsto q$, and g is a function vanishing at q, then it will pull back to a function vanishing at p." Note that locally ringed spaces form a category.

4.2. *Exercise.* Show that morphisms of locally ringed spaces glue (cf. Exercise 3.2). (Hint: Basically, the proof of Exercise 3.2 works.)

4.3. *Easy important exercise.* (a) Show that $\operatorname{Spec} R$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} R \to \operatorname{Spec} S$ defined by a ring morphism $f^\# : S \to R$ (Exercise 3.4) is a morphism of locally ringed spaces.

Proposition 4.1 now follows from:

4.4. *Key Proposition.* — *If* $f : \operatorname{Spec} R \to \operatorname{Spec} S$ *is a morphism of locally ringed spaces then it is the morphism of locally ringed spaces induced by the map* $f^\# : S = \Gamma(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \to \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R$. .

Proof. Suppose $f : \operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism of locally ringed spaces. Then we wish to show that $f^\# : \mathcal{O}_{\operatorname{Spec} S} \to f_* \mathcal{O}_{\operatorname{Spec} R}$ is the morphism of sheaves given by Exercise 3.4 (cf. Exercise 4.3(b)). It suffices to checked this on the distinguished base.

Note that if $s \in S$, $f^{-1}(D(s)) = D(f^\# s)$; this is where we use the hypothesis that f is a morphism of locally ringed spaces.

The commutative diagram

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) & \xrightarrow{f^{\#}_{\operatorname{Spec} S}} & \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \\ \downarrow & & \downarrow \otimes_S S_s \\ \Gamma(D(s), \mathcal{O}_{\operatorname{Spec} S}) & \xrightarrow{f^{\#}_{D(s)}} & \Gamma(D(f^{\#}s), \mathcal{O}_{\operatorname{Spec} R}) \end{array}$$

may be written as

$$\begin{array}{ccc} S & \xrightarrow{f^{\#}_{\operatorname{Spec} S}} & R \\ \downarrow & & \downarrow \otimes_S S_s \\ S_s & \xrightarrow{f^{\#}_{D(s)}} & R_{f^{\#}s}. \end{array}$$

We want that $f^{\#}_{D(s)} = (f^{\#}_{\operatorname{Spec} S})_s$. This is clear from the commutativity of that last diagram. □

In particular, we can check on an affine cover, and then we'll have it on all affines. Also, morphisms glue (Exercise 4.2). And: the composition of two morphisms is a morphism.

4.5. *Exercise.* Make sense of the following sentence: "$\mathbb{A}^{n+1} - \vec{0} \to \mathbb{P}^n$ given by

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto [x_0; x_1; \ldots; x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

4.6. The category of schemes (or k-schemes, or R-schemes, or Z-schemes).

We have thus defined a *category* of schemes. We then have notions of **isomorphism** and **automorphism**. It is often convenient to consider subcategories. For example, the category of k-schemes (where k is a field) is defined as follows. The objects are morphisms of the form $\begin{array}{c} X \\ \downarrow \\ \operatorname{Spec} k \end{array}$. (This is definition is identical to the one we gave earlier, but in a more satisfactory form.) The morphism (in the category of schemes, not in the category of k-schemes) $X \to \operatorname{Spec} k$ is called the **structure morphism**. The morphisms in the category of k-schemes are commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{=} & \operatorname{Spec} k \end{array}$$

which is more conveniently written as a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} k. & \end{array}$$

For example, complex geometers may consider the category of $\mathbb{C}$-schemes.

When there is no confusion, simply the top row of the diagram is given. More generally, if R is a ring, the category of R-schemes is defined in the same way, with R replacing k. And if Z is a scheme, the category of Z-schemes is defined in the same way, with Z replacing $\operatorname{Spec} k$.

E-mail address: vakil@math.stanford.edu