FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 17

CONTENTS

Last day: effective Cartier divisors; quasicoherent sheaves on projective A-schemes corresponding to graded modules, line bundles on projective A-schemes, O(n), generated by global sections, Serre's theorem, the adjoint functors $\sim$ and Γ_*.

Today: Associated points; more on normality; invertible sheaves and divisors take 1.

Our goal for today and part of next day is to develop tools to understand what invertible sheaves there can be on a scheme. As a key motivating example, we will show (by next day) that the only invertible sheaves on $\mathbb{P}^n_k$ are $\mathcal{O}(m)$.

But first, I want to tell you about *associated points* and the *ring of fractions* of a scheme. This topic isn't logically needed, but it is a description of the "most interesting points" of a scheme, where "all the action is".

1. ASSOCIATED POINTS

Recall that for an integral (= irreducible + reduced, by an earlier homework problem) scheme X, we have the notion of the *function field*, which is the stalk at the generic point. For any affine open subset $\operatorname{Spec} R$, we have that R is a subring of the function field.

It would be nice to generalize this to more general schemes, with possibly many components, and with nonreduced behavior.

The answer to this question is that on a "nice" (Noetherian) scheme, there are a finite number of points that will have similar information. (On a locally Noetherian scheme, we'll also have the notion of associated points, but there could be an infinite number of them.) I then drew a picture of a scheme with two components, one of which looked like a (reduced) line, and one of which was a plane, with some nonreduced behavior ("fuzz") along a line of it, and even more nonreduced behavior ("more fuzz") at a point of the line.

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I stated that the associated points are the generic points of the two components, plus the generic point of the line where this is fuzz, and the point where there is more fuzz.

We will define associated points of locally Noetherian schemes, and show the following important properties. You can skip the proofs if you want, but you should remember these properties.

(1) The generic points of the irreducible components are associated points. The other associated points are called *embedded points*.

(2) If X is reduced, then X has no embedded points. (This jibes with the intuition of the picture of associated points I described earlier.)

(3) If X is affine, say $X = \operatorname{Spec} R$ affine, then the natural map

$$R \to \prod_{\text{associated } \mathfrak{p}} R_{\mathfrak{p}} \tag{1}$$

is an injection. The primes corresponding to the associated points of R will be called *associated primes*. (In fact this is backwards; we will define associated primes first, and then define associated points.) The ring on the right of (1) is called the *ring of fractions*. If X is a locally Noetherian scheme, then the products of the stalks at the associated points will be called the *ring of fractions* of X. Note that if X is integral, this is the function field.

We define a *rational function* on a locally Noetherian scheme: it is an equivalence class. Any function defined on an open set containing all associated points is a rational function. Two such are considered the same if they agree on an open subset containing all associated points. If X is reduced, this is the same as requiring that they are defined on an open set of each of the irreducible components. A rational function has a maximal domain of definition, because any two actual functions on an open set (i.e. sections of the structure sheaf over that open set) that agree as "rational functions" (i.e. on small enough open sets containing associated points) must be the same function, by this fact **(3)**. We say that a rational function f is *regular* at a point p if p is contained in this maximal domain of definition (or equivalently, if there is some open set containing p where f is defined).

We similarly define *rational and regular sections of an invertible sheaf* $\mathcal{L}$ on a scheme X.

(4) A function is a zero divisor if and only if it vanishes at an associated point of $\operatorname{Spec} R$.

Okay, let's get down to business.

An ideal $I \subset A$ is *primary* if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$.

I like to interpret maximal ideals as "the quotient is a field", and prime ideals as "the quotient is an integral domain". We can interpret primary ideals similarly as "the quotient is not 0, and every zero-divisor is nilpotent".

1.1. Exercise. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, we say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*.

1.2. Exercise. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

1.3. Exercise (reality check). Find all the primary ideals in $\mathbb{Z}$. (Answer: (0) and (p^n).)

(Here is an unimportant side remark for experts; everyone else should skip this. Warning: a prime power need not be primary. An example is given in Atiyah-Macdonald, p. 51. $A = k[x, y, z]/(xy - z^2)$. then $\mathfrak{p} = (x, y)$ is prime but $\mathfrak{p}^2$ is not primary. Geometric hint that there is something going on: this is a ruling of a cone.)

A *primary decomposition* of an ideal $I \subset A$ is an expression of the ideal as a finite intersection of primary ideals.

$$I = \cap_{i=1}^n \mathfrak{q}_i$$

If there are "no redundant elements" (i.e. the $\sqrt{\mathfrak{q}_i}$ are all distinct, and for no i is $\mathfrak{q}_i \supset \cap_{j \neq i} \mathfrak{q}_j$), we say that the decomposition is *minimal*. Clearly any ideal with a primary decomposition has a minimal primary decomposition (using Exercise 1.2).

1.4. Exercise. Suppose A is a Noetherian ring. Show that every proper ideal $I \neq A$ has a primary decomposition. (Hint: Noetherian induction.)

1.5. Important Example. Find a minimal primary decomposition of (x^2, xy). (Answer: $(x) \cap (x^2, xy, y^n)$.)

In order to study these objects, we'll need a definition and a useful fact.

If $I \subset A$ is an ideal, and $x \in A$, then $(I : x) := \{a \in A : ax \in I\}$. (We will use this terminology only for this section.) For example, x is a *zero-divisor* if $(0 : x) \neq 0$.

1.6. Useful Exercise. (a) If $\mathfrak{p}$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup^n \mathfrak{p}_i$. Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n.)

1.7. *Theorem (Uniqueness of primary decomposition). — Suppose I has a minimal primary decomposition*

$$I = \cap_{i=1}^n \mathfrak{q}_i.$$

Then the $\sqrt{\mathfrak{q}_i}$ are precisely the prime ideals that are of the form

$$\sqrt{(I : x)}$$

for some $x \in A$. Hence this list of primes is independent of the decomposition.

These primes are called the *associated primes* of the ideal.

Proof. We make a very useful observation: for any $x \in A$,

$$(I : x) = (\cap \mathfrak{q}_i : x) = \cap(\mathfrak{q}_i : x),$$

from which

$$\sqrt{(I : x)} = \cap \sqrt{(\mathfrak{q}_i : x)} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j. \tag{2}$$

Now we prove the result.

Suppose first that $\sqrt{(I : x)}$ is prime, say $\mathfrak{p}$. Then $\mathfrak{p} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j$ by (2), and by Exercise 1.6(a), $\mathfrak{p} = \mathfrak{p}_j$ for some j.

Conversely, we find an x such that $\sqrt{(I : x)} = \sqrt{\mathfrak{q}_i}$ $(= \mathfrak{p}_i)$. Take $x \in \cap_{j \neq i} \mathfrak{q}_j - \mathfrak{q}_i$ (which is possible by minimality of the primary decomposition). Then by (2), we're done. □

If A is a ring, the *associated primes* of A are the associated primes of 0.

1.8. Exercise. Show that these associated primes behave well with respect to localization. In other words if A is a Noetherian ring, and S is a multiplicative subset (so, as we've seen, there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

We then define the *associated points* of a locally Noetherian scheme X to be those points $p \in X$ such that, on any affine open set $\operatorname{Spec} A$ containing p, p corresponds to an associated prime of A. If furthermore X is quasicompact (i.e. X is a Noetherian scheme), then there are a finite number of associated points.

1.9. Exercise. Show that the minimal primes of 0 are associated primes. (We have now proved important fact **(1)**.) (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^n \mathfrak{q}_i} = \cap_{i=1}^n \sqrt{\mathfrak{q}_i} = \cap_{i=1}^n \mathfrak{p}_i$, so by Exercise 1.6(b), $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \supset (0)$, we must have $\mathfrak{p} = \mathfrak{p}_i$.) Show that there can be other associated primes that are not minimal. (Hint: Exercise 1.5.)

1.10. Exercise. Show that if A is reduced, then the only associated primes are the minimal primes. (This establishes **(2)**.)

The $\mathfrak{q}_i$ corresponding to minimal primes are unique, but the $\mathfrak{q}_i$ corresponding to other associated primes are not unique, but we will not need this fact, and hence won't prove it.

1.11. *Proposition. — The natural map* $R \to \prod R_{\mathfrak{p}}$ *is an inclusion.*

This establishes **(3)**.

Proof. Suppose $r \mapsto 0$. Thus there are $s_i \in R - \mathfrak{p}$ with $s_i r = 0$. Then $I := (s_1, \dots, s_n)$ is an ideal consisting only of zero-divisors. Hence $I \subseteq \cap \mathfrak{p}_i$. Then $I \subset \mathfrak{p}_i$ for some i by Exercise 1.6(c), contradicting $s_i \notin \mathfrak{p}_i$. ☐

1.12. *Proposition. — The set of zero-divisors is precisely the union of the associated primes.*

This establishes **(4)**: a function is a zero-divisor if and only if it vanishes at an associated point. Thus (for a Noetherian scheme) a function is a zero divisor if and only if its zero locus contains one of a finite set of points.

You may wish to try this out on the example of Exercise 1.5.

Proof. If $\mathfrak{p}_i$ is an associated prime, then $\mathfrak{p}_i = \sqrt{(0 : x)}$ from the proof of Theorem 1.7, so $\cup \mathfrak{p}_i$ is certainly contained in the set D of zero-divisors.

For the converse, verify the inclusions and equalities (**Exercise**)

$$D = \cup_{x \neq 0} (0 : x) \subseteq \cup_{x \neq 0} \sqrt{(0 : x)} \subseteq D.$$

Hence

$$D = \cup_{x \neq 0} \sqrt{(0 : x)} = \cup_x \left(\cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j \right) \subseteq \cup \mathfrak{p}_j$$

using (2). ☐

(Note for experts from Kirsten and Joe: Let X be a locally Noetherian scheme, $x \in X$. Then x is an associated point of X if and only if every nonunit of $\mathcal{O}_{X,x}$ is a zero-divisor. Proof: We must show that a prime ideal $\mathfrak{p}$ of a Noetherian ring A is associated if and only if every nonunit of $A_{\mathfrak{p}}$ is a zero-divisor, i.e., if and only if $\mathfrak{p}A_{\mathfrak{p}}$ is an associated prime in $A_{\mathfrak{p}}$. But this is obvious since primary decompositions respect localization.)

2. Invertible sheaves and divisors

We want to understand invertible sheaves (line bundles) on a given sheaf X. How can we describe many of them? How can we describe them all?

In order to answer this question, I should tell you a bit more about normality.

2.1. **A bit more on normality.**

I earlier defined normality in the wrong way, only for integral schemes: I said that an integral scheme X is normal if and only if for every affine open set $\operatorname{Spec} R$, R is integrally closed in its fraction field.

Here is the right definition: we say a scheme X is normal if all of its stalks $\mathcal{O}_{X,x}$ are normal. (In particular, all stalks are necessarily domains.) This is clearly a local property: if $\cup U_i$ is an open cover of X, then X is normal if and only if each U_i is normal.

Note that for Noetherian schemes, normality can be checked at closed points, as integral closure behaves well under localization (we've checked that), and every open set contains closed points of the scheme (we've checked that), so any point is a generization of a closed point.

As reducedness is a stalk-local property (we've checked that X is reduced if and only if all its stalks are reduced), a normal scheme is necessarily reduced. It is not true however that normal schemes are integral. For example, the disjoint union of two normal schemes is normal. So for example $\operatorname{Spec} k \coprod \operatorname{Spec} k \cong \operatorname{Spec}(k \times k) \cong \operatorname{Spec} k[x]/(x(x-1))$ is normal, but its ring of global sections is not a domain.

Unimportant remark. Normality satisfies the hypotheses of the Affine Covering Lemma, fairly tautologically, because it is a stalk-local property. We can say more explicitly and ring-theoretically what it means for $\operatorname{Spec} A$ to be normal, at least when A is Noetherian. It is that $\operatorname{Spec} A$ is normal if and only if A is reduced, and it is integrally closed in its ring of fractions. (The ring of fractions was defined earlier today in the discussion on associated points. It is the product of the localizations at the associated points. In this case, as A is reduced, it is the product of the localizations at the minimal primes.) Basically, most constructions that make sense for domains and involve function fields should be generalized to Noetherian rings in general, and the role of "function field" should be replaced by "ring of fractions".

I should finally state "Hartogs' theorem" explicitly and rigorously. (Caution: No one else calls this Hartogs' Theorem. I've called it this because of the metaphor to complex geometry.)

2.2. *"Hartogs' theorem". — Suppose A is a Noetherian normal domain. Then in $\operatorname{Frac}(A)$,*

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

More generally, if A is a product of Noetherian normal domains (i.e. $\operatorname{Spec} A$ is Noetherian normal scheme), then in the ring of fractions of A,

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

I stated the special case first so as to convince you that this isn't scary.

To show you the power of this result, let me prove Krull's Principal Ideal Theorem in the case of Noetherian normal domains. (Eventually, I hope to add to the notes a proof of Krull's Principal Ideal Theorem in general, as well as "Hartogs' Theorem".)

2.3. *Theorem (Krull's Principal Ideal Theorem for Noetherian normal domains). — Suppose A is a Noetherian normal domains, and $f \in A$. Then the minimal primes containing f are all of height precisely 1.*

Proof. The first statement implies the second: because A is a domain, the associated primes of $\operatorname{Spec} A$ are precisely the minimal (i.e. height 0) primes. If f is a not a zero-divisor, then f is not an element of any of these primes, by Proposition 1.12.

So we will now prove the first statement.

Suppose $f \in \mathrm{Frac}(A)$. We wish to show that the minimal primes containing f are all height 1. If there is one which is height greater than 1, then after localizing at this prime, we may assume that A is a local ring with maximal ideal $\mathfrak{m}$ of height at least 2, and that the only prime containing f is $\mathfrak{m}$. Let $g = 1/f \in \mathrm{Frac}(A)$. Then $g \in A_{\mathfrak{p}}$ for all height 1 primes $\mathfrak{p}$, so by "Hartogs' Theorem", $g \in A$. Thus $gf = 1$. But $g, f \in A$, and $f \in \mathfrak{m}$, so we have a contradiction.

Exercise. Suppose f and g are two global sections of a Noetherian normal scheme with the same poles and zeros. Show that each is a unit times the other.

I spent the rest of the class discussing Cartier divisors. I've put these notes with the class 18 notes.

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