FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 9

CONTENTS

Last day: irreducible, connected, quasicompact, reduced, dimension.

Today: Krull's Principal Ideal Theorem, height, affine communication lemma, properties of schemes: locally Noetherian, Noetherian, finite type S-scheme, locally of finite type S-scheme, normal.

I realize now that you may not have seen the notion of transcendence degree. I'll tell you the main thing you need to know about it, which I hope you will find believable. Suppose K/k is a finitely generated field extension. Then any two maximal sets of algebraically independent elements of K over k (i.e. any set with no algebraic relation) have the same size (a non-negative integer or ∞). If this size is finite, say n, and $x_1, \ldots, x_n$ is such a set, then $K/k(x_1, \ldots, x_n)$ is necessarily a finitely generated algebraic extension, i.e. a finite extension. (Such a set $x_1, \ldots, x_n$ is called a transcendence basis, and n is called the *transcendence degree*.) A short and well-written proof of this fact is in *Mumford's Red Book of Varieties and Schemes*.

1. DIMENSION, CONTINUED

Last day, I defined the dimension of a scheme. I defined the dimension (or Krull dimension) as the supremum of lengths of chains of closed irreducible sets, starting indexing with 0. This dimension is allowed to be ∞. For example: a Noetherian topological space has a finite dimension. The Krull dimension of a ring is the Krull dimension of its topological space. It is one less than the length of the longest chain of nested prime ideals you can find.

We are in the midst of proving the following result, which lets us understand dimension when working in good situations.

1.1. *Big Theorem of last day. — Suppose R is a finitely-generated domain over a field k. Then $\dim \operatorname{Spec} R$ is the transcendence degree of the fraction field $\operatorname{Frac}(R)$ over k.*

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1.2. Exercise. Suppose X is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where R is a finitely generated domain over k. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X.

The proof of the big theorem will rely on two different facts pulling in opposite directions. The first is the following lemma, which we proved.

1.3. *Lemma. — Suppose R is an integral domain over k (not necessarily finitely generated, although that is the case we will care most about), and $\mathfrak{p} \subset R$ a prime. Then $\dim_{tr} R \geq \dim_{tr} R/P$, with equality if and only $\mathfrak{p} = (0)$, or $\dim_{tr} R/P = \infty$.*

You should have a picture in your mind when you hear this: if you have an irreducible space of finite dimension, then any proper subspace has strictly smaller dimension — certainly believable!

This implies that $\boxed{\dim R \leq \dim_{tr} R}$. (Think this through!)

The other fact we'll use is Krull's Principal Ideal Theorem. This result is one of the few hard facts I'll not prove. We may prove it later in the class (possibly in a problem set), and you can also read a proof in Mumford's *Red Book*, in §I.7, where you'll find much of this exposition.

1.4. *Krull's Principal Ideal Theorem (transcendence degree version). — Suppose R is a finitely generated domain over k, $f \in R$, $\mathfrak{p}$ a minimal prime of R/f. Then if $f \neq 0$, $\dim_{tr} R/\mathfrak{p} = \dim_{tr} R - 1$.*

This is best understood geometrically: if you have some irreducible space of finite dimension, then any non-zero function on it cuts out a set of *pure* codimension 1. Somewhat more precisely:

1.5. *Theorem (geometric interpretation of Krull). — Suppose $X = \operatorname{Spec} R$ where R is a finitely generated domain over k, $g \in R$, Z an irreducible component of $V(g)$. Then if $g \neq 0$, $\dim_{tr} Z = \dim_{tr} X - 1$.*

Before I get to the proof of the theorem, I want to point out that this is useful on its own. Consider the scheme $\operatorname{Spec} k[w,x,y,z]/(wx - yz)$. What is its dimension? It is cut out by one non-zero equation $wx - yz$ in $\mathbb{A}^4$, so it is a threefold.

1.6. Exercise. What is the dimension of $\operatorname{Spec} k[w,x,y,z]/(wx - yz, x^{17} + y^{17})$? (Be careful to check they hypotheses before invoking Krull!)

1.7. Exercise. Show that $\operatorname{Spec} k[w,x,y,z]/(wz - xy, wy - x^2, xz - y^2)$ is an integral *surface*. You might expect it to be a curve, because it is cut out by three equations in 4-space. (Remark for experts: this is not a random ideal. In language we will later make precise: it is the affine cone over a curve in $\mathbb{P}^3$. This curve is called the *twisted cubic*. It is in some sense the simplest curve in $\mathbb{P}^3$ not contained in a hyperplane. You can think of it as the points of the form (t, t^2, t^3) in $\mathbb{A}^3$. Indeed, you'll notice that $(w, x, y, z) = (a, at, at^2, at^3)$ satisfies the equations above. It turns out that you actually need three equations to cut out this surface. The first equation cuts out a threefold in four-space (by Krull's theorem, see later). The second equation cuts out a surface: our surface, and another surface. The third equation cuts out our surface. One last aside: notice once again that the cone over the quadric surface $k[w, x, y, z]/(wz - xy)$ makes an appearance.)

We'll now put together our lemma, and this geometric interpretation of Krull. Notice the interplay between the two: the first says that the dimension definitely drops when you take a proper irreducible closed subset. The second says that you can arrange for it to drop by precisely 1.

I proved the following result, which I didn't end up using.

1.8. *Proposition.* — *Suppose* X *is the* Spec *of a finitely generated domain over* k, *and* Z *is an irreducible closed subset, maximal among all proper irreducible closed subsets of* X. *(I gave a picture here.) Then* $\dim_{tr} Z = \dim_{tr} X - 1$.

(We certainly have $\dim_{tr} Z \leq \dim_{tr} X - 1$ by our lemma.)

Proof. Suppose $Z = V(\mathfrak{p})$ where $\mathfrak{p}$ is prime. Choose any non-zero $g \in \mathfrak{p}$. By Krull's theorem, the components of $V(g)$ are have $\dim_{tr} = \dim_{tr} X - 1$. Z is contained in one of the components. By the maximality of Z, Z *is* one of the components. $\square$

1.9. *Proof of big theorem.* We prove it by induction on $\dim_{tr} X$. The base case $\dim_{tr} X = 0$ is easy: by our lemma, $\dim X \leq \dim_{tr} X$, so $\dim X = 0$.

Now assume that $\dim_{tr} X = n$. As $\dim X \leq \dim_{tr} X$, our goal will be to produce a chain of $n + 1$ irreducible closed subsets. Say $X = \operatorname{Spec} R$. Choose any $g \neq 0$ in R. Choose any component Z of $V(g)$. Then $\dim_{tr} Z = n - 1$ by Krull's theorem, and the inductive hypothesis, so we can find a chain of n irreducible closed subsets descending from Z. We're done. $\square$

I gave a geometric picture of both. Note that equality needn't hold in the first case.

The big theorem is about the dimension of finitely generated domains over k. For such rings, dimension is well-behaved. This set of rings behaves well under quotients; I want to show you that it behaves well under localization as well.

1.10. *Proposition.* — *Suppose* R *is a finitely generated domain over* k, *and* $\mathfrak{p}$ *is a prime ideal. Then* $\dim R_{\mathfrak{p}} = \dim R - \dim R/\mathfrak{p}$.

The scheme-theoretic version of this statement about rings is: $\dim_{\mathcal{O}_{Z,X}} = \dim X - \dim Z$.

1.11. Exercise. Prove this. (I gave a geometric explanation of why this is true, which you can take as a "hint" for this exercise.) In the course of this exercise, you will show the important fact that if $n = \dim R$, then any chain of prime ideals can be extended to a chain of prime ideals of length n. Further, given a prime ideal, you can tell where it is in any chain by looking at the transcendence degree of its quotient field. This is a particularly nice feature of polynomial rings, that will not hold even for Noetherian rings in general (see the next section).

2. Height, and Krull's Principal Ideal Theorem

This is a good excuse to tell you a definition in algebra. Definition: the *height* of the prime ideal $\mathfrak{p}$ in R is $\dim R_{\mathfrak{p}}$. Algebraic translation: it is the supremum of lengths of chains of primes contained in $\mathfrak{p}$.

This is a good but imperfect version of codimension. For finitely generated domains over k, the two notions agree, by Proposition 1.10. An example of a pathology is given below.

With this definition of height, I can state a more general version of Krull's Principal Ideal Theorem.

2.1. *Krull's Principal Ideal Theorem. — Suppose R is a Noetherian ring, and $f \in A$ an element which is not a zero divisor. Then every minimal prime $\mathfrak{p}$ containing f has height 1.* (Atiyah-Macdonald p. 122)

(We could have $V(f) = \emptyset$, if f is a unit — but that doesn't violate the statement.)

The geometric picture is the same as before: "If f is not a zero-divisor, the codimension is 1."

It is possible that I will give a proof later in the course. Either I'll give an algebraic proof in the notes, or I will give a geometric proof in class, using concepts we have not yet developed. (I'll be careful to make sure the argument is not circular!)

2.2. Important Exercise. (This will be useful soon.) (a) Suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible component of $V(r_1, \dots, r_n)$, where $r_1, \dots, r_n \in R$. Show that the height of (the prime associated to) Z is at most n. Conversely, suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible subset of height n. Show that there are $f_1, \dots, f_n \in R$ such that Z is an irreducible component of $V(f_1, \dots, f_n)$.
(b) (application to finitely generated k-algebras) Suppose $X = \operatorname{Spec} R$ where R is a finitely generated domain over k, and Z is an irreducible component of $V(r_1, \dots, r_n)$, where $r_1, \dots, r_n \in R$. Show that $\dim Z \geq \dim X - n$. Conversely, suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible subset of codimension n. Show that there are $f_1, \dots, f_n \in R$ such that Z is an irreducible component of $V(f_1, \dots, f_r)$.

2.3. Important but straightforward exercise. If R is a finitely generated domain over k, show that $\dim R[x] = \dim R + 1$. If R is a Noetherian ring, show that $\dim R[x] \geq \dim R + 1$. (Fact, proved later: if R is a Noetherian ring, then $\dim R[x] = \dim R + 1$. We'll prove this later. You may use this fact in exercises in later weeks.)

We now show how the height can behave badly. Let $R = k[x]_{(x)}[t]$. In other words, elements of R are polynomials in t, whose coefficients are quotients of polynomials in x, where no factors of x appear in the denominator. R is a domain. $(xt-1)$ is not a zero divisor. You can verify that $R/(xt-1) \cong k[x]_{(x)}[1/x] \cong k(x)$ — "in $k[x]_{(x)}$, we may divide by everything but x, and now we are allowed to divide by x as well" — so $R/(xt-1)$ is a field. Thus $(xt-1)$ is not just prime but also maximal. By Krull's theorem, $(xt-1)$ is height 1. Thus $(0) \subset (xt-1)$ is a maximal chain. However, R has dimension at least 2: $(0) \subset (t) \subset (x,t)$ is a chain of primes of length 3. (In fact, R has dimension precisely 2: $k[x]_{(x)}$ has dimension 1, and the fact mentioned in the previous exercise 2.3 implies $\dim k[x]_{(x)}[t] = \dim k[x]_{(x)} + 1 = 2$.) Thus we have a height 1 prime in a dimension 2 ring that is "codimension 2". A picture of this lattice of ideals is below.

$$
\begin{tikzcd}[column sep=small]
(x,t) \arrow[d, dash] & & \\
(t) \arrow[dr, dash] & & (xt-1) \arrow[dl, dash] \\
& (0) &
\end{tikzcd}
$$

(This example comes from geometry; it is enlightening to draw a picture. $k[x]_{(x)}$ corresponds to a germ of $\mathbb{A}^1_k$ near the origin, and $k[x]_{(x)}[t]$ corresponds to "this $\times$ the affine line".) For this reason, codimension is a badly behaved notion in Noetherian rings in general.

I find it disturbing that this misbehavior turns up even in a relative benign-looking ring.

3. Properties of schemes that can be checked "affine-locally"

Now I want to describe a host of important properties of schemes. All of these are "affine-local" in that they can be checked on any affine cover, by which I mean a covering by open affine sets.

Before I get going, I want to point out something annoying in the definition of schemes. I've said that a scheme is a topological space with a sheaf of rings, that can be covered by affine schemes. There is something annoying about this description that I find hard to express. We have all these affine opens in the cover, but we don't know how to communicate between any two of them. Put a different way, if I have an affine cover, and you have an affine cover, and we want to compare them, and I calculate something on my cover, there should be some way of us getting together, and figuring out how to translate my calculation over onto your cover. (I'm not sure if you buy what I'm trying to sell here.) The affine communication lemma I'll soon describe will do this for us.

3.1. *Remark.* In our limited examples so far, any time we've had an affine open subset of an affine scheme $\operatorname{Spec} S \subset \operatorname{Spec} R$, in fact $\operatorname{Spec} S = D(f)$ for some f. But this is not always true, and we will eventually have an example. (We'll first need to define elliptic curves!)

3.2. *Proposition. — Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* X. *Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distinguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$.

Proof. (This is best seen with a picture, which unfortunately won't be in the notes.) Given any $\mathfrak{p} \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we produce an open neighborhood of $\mathfrak{p}$ in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is simultaneously distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f$. Then $g \in \Gamma(\operatorname{Spec} B, \mathcal{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X) = A_f$. The points of $\operatorname{Spec} A_f$ where g vanishes are precisely the points of $\operatorname{Spec} A_f$ where g' vanishes (cf. earlier exercise), so

$$\begin{aligned} \operatorname{Spec} B_g &= \operatorname{Spec} A_f \setminus \{\mathfrak{p} : g' \in \mathfrak{p}\} \\ &= \operatorname{Spec}(A_f)_{g'}. \end{aligned}$$

If $g' = g''/f^n$ ($g'' \in A$) then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$, and we are done. □

3.3. *Affine communication lemma. — Let* P *be some property enjoyed by some affine open sets of a scheme* X, *such that*

(i) *if* $\operatorname{Spec} R \hookrightarrow X$ *has* P *then for any* $f \in R$, $\operatorname{Spec} R_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \dots, f_n) = R$, *and* $\operatorname{Spec} R_{f_i} \hookrightarrow X$ *has* P *for all* i, *then so does* $\operatorname{Spec} R \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} R_i$ *where* $\operatorname{Spec} R_i$ *is an affine, and* R_i *has property* P. *Then every other open affine subscheme of* X *has property* P *too.*

Proof. (This is best done with a picture.) Cover $\operatorname{Spec} R$ with a finite number of distinguished opens $\operatorname{Spec} R_{g_j}$, each of which is distinguished in some R_{f_i}. This is possible by Proposition 3.2 and the quasicompactness of $\operatorname{Spec} R$. By (i), each $\operatorname{Spec} R_{g_j}$ has P. By (ii), $\operatorname{Spec} R$ has P. □

By choosing P appropriately, we define some important properties of schemes.

3.4. *Proposition. — Suppose* R *is a ring, and* $(f_1, \dots, f_n) = R$.

(a) *If* R *is a Noetherian ring, then so is* R_{f_i}. *If each* R_{f_i} *is Noetherian, then so is* R.
(b) *If* R *has no nonzero nilpotents (i.e.* 0 *is a radical ideal), then* R_{f_i} *also has no nonzero nilpotents. If no* R_{f_i} *has a nonzero nilpotent, then neither does* R. **Do we say "a ring is reduced? radical?"**

(c) *Suppose* A *is a ring, and* R *is an* A*-algebra. If* R *is a finitely generated* A*-algebra, then so is* R_{f_i}*. If each* R_{f_i} *is a finitely-generated* A*-algebra, then so is* R*. (I didn't say this in class, so I'll say it on Monday.)*
(d) *Suppose* R *is an integral domain. If* R *is integrally closed, then so is* R_{f_i}*. If each* R_{f_i} *is integrally closed, then so is* R*.*

We'll prove these shortly. But given this, I want to make some definitions.

3.5. Important Definitions. Suppose X is a scheme.

- If X can be covered by affine opens $\operatorname{Spec} R$ where R is Noetherian, we say that X is a *locally Noetherian scheme*. If in addition X is quasicompact, or equivalently can be covered by finitely many such affine opens, we say that X is a *Noetherian scheme* **Exercise.** Show that the underlying topological space of a Noetherian scheme is Noetherian. **Exercise.** Show that all open subsets of a Noetherian scheme are quasicompact.
- If X can be covered by affine opens $\operatorname{Spec} R$ where R is reduced (nilpotent-free), we say that X is *reduced*. **Exercise:** Check that this agrees with our earlier definition. This definition is advantageous: our earlier definition required us to check that the ring of functions over *any* open set is nilpotent free. This lets us check in an affine cover. Hence for example $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are reduced.
- Suppose A is a ring (e.g. A is a field k), and $\Gamma(X, \mathcal{O}_X)$ is an A-algebra. Then we say that X is an A*-scheme*, or a *scheme over* A. Suppose X is an A-scheme. (Then for any non-empty U, $\Gamma(U, \mathcal{O}_X)$ is naturally an A-algebra.) If X can be covered by affine opens $\operatorname{Spec} R$ where R is a *finitely generated* A-algebra, we say that X is *locally of finite type over* A, or that it is a *locally of finite type* A*-scheme*. (My apologies for this cumbersome terminology; it will make more sense later.) If furthermore X is quasicompact, X is *finite type over* A, or a *finite type* A*-scheme*.
- If X is integral, and can be covered by affine opens $\operatorname{Spec} R$ where R is a integrally closed, we say that X is *normal*. (Thus in my definition, normality can only apply to integral schemes. I may want to patch this later.) **Exercise.** If R is a unique factorization domain, show that $\operatorname{Spec} R$ is integrally closed. Hence $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are both normal.

Proof. (a) (i) If $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of R_f, then we can verify that $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of R, where

$$J_j = \{r \in R : r \in I_j\}$$

where $r \in I_j$ means "the image in R_f lies in I_j". (We think of this as $I_j \cap R$, except in general R needn't inject into R_{f_i}.) Clearly J_j is an ideal of R. If $x/f^n \in I_{j+1} \setminus I_j$ where $x \in R$, then $x \in J_{j+1}$, and $x \notin J_j$ (or else $x(1/f)^n \in J_j$ as well). (ii) Suppose $I_1 \subsetneq I_2 \subsetneq I_3 \subset \cdots$ is a strictly increasing chain of ideals of R. Then for each $1 \leq i \leq n$,

$$I_{i,1} \subset I_{i,2} \subset I_{i,3} \subset \cdots$$

is an increasing chain of ideals in R_{f_i}, where $I_{i,j} = I_j \otimes_R R_{f_i}$. We will show that for each j, $I_{i,j} \subsetneq I_{i,j+1}$ for some i; the result will then follow.

(b) **Exercise.**

(c) (I'll present this on Monday.) (i) is clear: if R is generated over S by $r_1, \dots, r_n$, then R_f is generated over S by $r_1, \dots, r_n, 1/f$.

(ii) Here is the idea; I'll leave this as an **exercise** for you to make this work. We have generators of R_i: r_{ij}/f_i^j, where $r_{ij} \in R$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_j$ generate R as a S-algebra. Here's why. Suppose you have any $r \in R$. Then in R_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each R_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the R_{f_i}. Thus it is indeed r.

(d) (i) is easy. If $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in R_f$ has a root in the fraction field. Then we can easily show that the root lies in R_f, by multiplying by enough f's to kill the denominator, then replacing $f^a x$ by y. That is likely incomprehensible, so I'll leave this as an **exercise**.

(ii) (This one involves a neat construction.) Suppose R is not integrally closed. We show that there is some f_i such that R_{f_i} is also not integrally closed. Suppose

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{1}$$

(with $a_i \in R$) has a solution s in $\operatorname{Frac}(R)$. Let I be the "ideal of denominators" of s:

$$I := \{r \in R : rs \in R\}.$$

(Note that I is clearly an ideal of R.) Now $I \neq R$, as $1 \notin I$. As $(f_1, \dots, f_n) = R$, there must be some $f_i \notin I$. Then $s \notin R_{f_i}$, so equation (1) in $R_{f_i}[x]$ shows that R_{f_i} is not integrally closed as well, as desired. $\square$

3.6. *Unimportant Exercise relating to the proof of (d).* One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $R = k[a, b, c, d]/(ad - bc)$, and $a/c = b/d \in \operatorname{Frac}(R)$. Then it turns out that $I = (c, d)$, which is not principal. We'll likely show that it is not principal at the start of the second quarter. (I could give a one-line explanation right now, but this topic makes the most sense when we talk about Zariski tangent spaces.)

E-mail address: `vakil@math.stanford.edu`