FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 8

CONTENTS

Last day: $\tilde{M}$, sheaf associated to R-module M; Chinese remainder theorem; germs and value at a point of the structure sheaf; non-affine schemes $\mathbb{A}^2 - (0,0)$, line with doubled origin, $\mathbb{P}^n$.

Today: irreducible, connected, quasicompact, reduced, dimension.

1. PROPERTIES OF SCHEMES

We're now going to define properties of schemes. We'll start with some topological properties.

I've already defined what it means for a topological space to be *irreducible*: if X is the union of two closed subsets $U \cup V$, then either $X = U$ or $X = V$.

Problem A4 on problem set 3 implies that $\mathbb{A}^n_k$ is irreducible. There is a one-line answer. This argument "behaves well under gluing", yielding:

1.1. Exercise. Show that $\mathbb{P}^n_k$ is irreducible.

1.2. Exercise. You showed earlier that for affine schemes, there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

In the examples we have considered, the spaces have naturally broken up into some obvious pieces. Let's make that a bit more precise.

A topological space X is called *Noetherian* if it satisfies the descending chain condition for closed subsets: any sequence $Z_1 \supseteq Z_2 \supset \cdots$ of closed subsets eventually stabilizes: there is an r such that $Z_r = Z_{r+1} = \cdots$.

I showed some examples on $\mathbb{A}^2$, to show that it can take arbitrarily long to stabilize.

Date: Monday, October 24, 2005. Small correction January 31, 2007.

All of the cases we have considered have this property, but that isn't true of all rings. The key characteristic all of our examples have had in common is that the rings were *Noetherian*. Recall that a ring is *Noetherian* if *ascending* sequence $I_1 \subset I_2 \subset \cdots$ of closed ideals eventually stabilizes: there is an r such that $I_r = I_{r+1} = \cdots$.

Here are some quick facts about Noetherian rings. You should be able to prove them all.

- Fields are Noetherian. $\mathbb{Z}$ is Noetherian.
- If R is Noetherian, and I is any ideal, then R/I is Noetherian.
- If R is Noetherian, and S is any multiplicative set, then $S^{-1}R$ is Noetherian.
- In a Noetherian ring, any ideal is finitely generated. Any submodule of a finitely generated module over a Noetherian ring is finitely generated.

The next fact is non-trivial.

1.3. *The Hilbert basis theorem. — If* R *is Noetherian, then so is* $R[x]$.

Proof omitted. (This was done in Math 210.)

(I then discussed the game of Chomp. The fact that the game of infinite chomp is guaranteed to end is an analog of the Hilbert basis theorem. In fact, this is a consequence of the Hilbert basis theorem — the fact that infinite chomp is guaranteed to end corresponds to the fact that any ascending chain of monomial ideals in $k[x, y]$ must eventually stabilize. I learned of this cute fact from Rahul Pandharipande. If you prove the Chomp problem, you'll understand how to prove the Hilbert basis theorem.)

Using these results, then any polynomial ring over any field, or over the integers, is Noetherian — and also any quotient or localization thereof. Hence for example any finitely-generated algebra over k or $\mathbb{Z}$, or any localization thereof is Noetherian.

1.4. Exercise. Prove the following. If R is Noetherian, then $\operatorname{Spec} R$ is a Noetherian topological space. If X is a scheme that has a finite cover $X = \cup_{i=1}^{n} \operatorname{Spec} R_i$ where R_i is Noetherian, then X is a Noetherian topological space.

Thus $\mathbb{P}_k^n$ and $\mathbb{P}_{\mathbb{Z}}^n$ are Noetherian topological spaces: we built them by gluing together a finite number of Spec's of Noetherian rings.

If X is a topological space, and Z is an irreducible closed subset not contained in any larger irreducible closed subset, Z is said to be an *irreducible component* of X. (I drew a picture.)

1.5. Exercise. If R is any ring, show that the irreducible components of $\operatorname{Spec} R$ are in bijection with the minimal primes of R. (Here minimality is with respect to inclusion.)

For example, the only minimal prime of $k[x, y]$ is (0). What are the minimal primes of $k[x, y]/(xy)$?

1.6. *Proposition. — Suppose* X *is a Noetherian topological space. Then every non-empty closed subset* Z *can be expressed uniquely as a finite union* $Z = Z_1 \cup \cdots \cup Z_n$ *of irreducible closed subsets, none contained in any other.*

As a corollary, this implies that a Noetherian ring R has only finitely many minimal primes.

Proof. The following technique is often called *Noetherian induction*, for reasons that will become clear.

Consider the collection of closed subsets of X that *cannot* be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Otherwise, let Y_1 be one such. If it properly contains another such, then choose one, and call it Y_2. If this one contains another such, then choose one, and call it Y_3, and so on. By the descending chain condition, this must eventually stop, and we must have some Y_r that cannot be written as a finite union of irreducible closed subsets, but every closed subset contained in it can be so written. But then Y_r is not itself irreducible, so we can write $Y_r = Y' \cup Y''$ where Y' and Y'' are both proper closed subsets. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can Y_r, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them.

We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z_1' \cup Z_2' \cup \cdots \cup Z_s'$$

are two such representations. Then $Z_1' \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_r \cap Z_1')$. Now Z_1' is irreducible, so one of these is Z_1' itself, say (without loss of generality) $Z_1 \cap Z_1'$. Thus $Z_1' \subset Z_1$. Similarly, $Z_1 \subset Z_a'$ for some a; but because $Z_1' \subset Z_1 \subset Z_a'$, and Z_1' is contained in no other Z_i', we must have $a = 1$, and $Z_1' = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa, so they must be the same list. □

1.7. Connectedness and quasicompactness.

Definition. A topological space X is connected if it cannot be written as the disjoint union of two non-empty open sets.

We say that a subset Y of X is a *connected component* if it is connected, and both open and closed. **Remark added later: Thanks to Anssi for pointing out that this is not the usual definition of connected component. The usual definition, which deals with more pathological situations, implies this one. At some point I might update these notes and say more.**

1.8. Exercise. Show that an irreducible topological space is connected.

1.9. Exercise. Give (with proof!) an example of a scheme that is connected but reducible.

We have already defined **quasicompact**.

1.10. Exercise. Show that a finite union of affine schemes is quasicompact. (Hence $\mathbb{P}^n_k$ is quasicompact.) Show that every closed subset of an affine scheme is quasicompact. Show that every closed subset of a quasicompact scheme is quasicompact.

The last topological property I should discuss is *dimension*. But that will take me some time, and it will involve some non-topological issues, so I'll first talk about an important non-topological property. Remember that one of the alarming things about schemes is that functions are not determined by their values at points, and that was because of the presence of *nilpotents*.

1.11. Definition. We will say that a ring is *reduced* if it has no nilpotents. A scheme is *reduced* if $\mathcal{O}_X(U)$ has no nonzero nilpotents for any open set U of X.

An example of a nonreduced affine scheme is $k[x,y]/(xy,x^2)$. Picture: y-axis with some fuzz at the origin (I drew this). The fuzz indicates that there is some nonreducedness going on at the origin. Here are two different functions: y and $x+y$. Their values agree at all points. They are actually the same function on the open set $D(y)$, which is not surprising, as $D(y)$ is reduced, as the next exercise shows.

1.12. Exercise. Show that $\left(k[x,y]/(xy,x^2)\right)_y$ has no nilpotents. (Hint: show that it is isomorphic to another ring, by considering the geometric picture.)

1.13. Exercise. Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$.

Definition. A scheme is *integral* if $\mathcal{O}_X(U)$ is an integral domain for each open set U of X.

1.14. Exercise. Show that an affine scheme $\operatorname{Spec} R$ is integral if and only if R is an integral domain.

1.15. Exercise. Show that a scheme X is integral if and only if it is irreducible and reduced.

1.16. Exercise. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X. Show that the stalk at η, $\mathcal{O}_{X,\eta}$, is naturally $\operatorname{Frac} R$. This is called the *function field* of X. It can be computed on any non-empty open set of X (as any such open set contains the generic point).

1.17. Exercise. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \emptyset$. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X (so R is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \operatorname{Frac} R$ (where U is any non-empty open set) is an inclusion.

2. Dimension

Our goal is to define the dimension of schemes. This should agree with, and generalize, our geometric intuition. (Careful: if you are thinking over the complex numbers, our dimensions will be complex dimensions, and hence half that of the "real" picture.) We will also use it to prove things; as a preliminary example, we will classify the prime ideals of $k[x, y]$.

It turns out that the right definition is purely topological — it just depends on the topological space, and not at all on the structure sheaf. Define **dimension** by Krull dimension: the supremum of lengths of chains of closed irreducible sets, starting indexing with 0. This dimension is allowed to be ∞. Define the Krull dimension of a ring to be the Krull dimension of its topological space. It is one less than the length of the longest chain of nested prime ideals you can find. (You might think a Noetherian ring has finite dimension, but this isn't necessarily true. For a counterexample by Nagata, who is the master of all counterexamples, see Eisenbud's *Commutative Algebra with a View to Algebraic Geometry*, p. 231.)

(Scholars of the empty set can take the dimension of the empty set to be $-\infty$.)

Obviously the Krull dimension of a ring R is the same as the Krull dimension of $R/\mathfrak{N}$: dimension doesn't care about nilpotents.

For example: We have identified the prime ideals of $k[t]$, so we can check that $\dim \mathbb{A}^1 = 1$. Similarly, $\dim \operatorname{Spec} \mathbb{Z} = 1$. Also, $\dim \operatorname{Spec} k = 0$, and $\dim \operatorname{Spec} k[x]/x^2 = 0$.

Caution: if Z is the union of two closed subsets X and Y, then $\dim_Z = \max(\dim X, \dim Y)$. In particular, if Z is the disjoint union of something of dimension 2 and something of dimension 1, then it has dimension 2. Thus dimension is not a "local" characteristic of a space. This sometimes bothers us, so we will often talk about dimensions of irreducible topological spaces. If a topological space can be expressed as a finite union of irreducible subsets, then say that it is *equidimensional* or *pure dimensional* (resp. equidimensional of dimension n or pure dimension n) if each of its components has the same dimension (resp. they are all of dimension n).

The notion of *codimension* of something equidimensional in something equidimensional makes good sense (as the difference of the two dimensions). Caution (added Nov. 6): there is another possible definition of codimension, in terms of *height*, defined later. Hartshorne uses this second definition. These two definitions can disagree — see e.g. the example of "height behaving badly" in the Class 9 notes. So we will be very cautious in using then word "codimension".

An equidimensional dimension 1 (resp. 2, n) topological space is said to be a *curve* (resp. surface, n-fold).

2.1. Reality check. Show that $\dim R/\mathfrak{p} \leq \dim R$, where $\mathfrak{p}$ is prime. Hope: equality holds if and only if $\mathfrak{p} = 0$ or $\dim R/\mathfrak{p} = \infty$. It is immediate that if R is a finite-dimensional domain, and $\mathfrak{p} \neq 0$, then we have inequality.

Warning: in all of the examples we have looked at, they behave well, but dimension can behave quite pathologically. But in good situations, including ones that come up more naturally in nature, it doesn't. For example, in cases involving a finite number variables over a field, dimension follows our intuition. More precisely:

2.2. *Big Theorem of today.* — Suppose R is a finitely-generated domain over a field k. Then $\dim \operatorname{Spec} R$ is the transcendence degree of the fraction field $\operatorname{Frac}(R)$ over k.

(By "finitely generated domain over k", I mean "a finitely generated k-algebra that is an integral domain". I'm just trying to save ink.)

Note that these finitely generated domains over k can each be described as the ring of functions on an irreducible subset of some $\mathbb{A}^n$: given such a domain, choose generators $x_1, \ldots, x_n$. Conversely, if $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ is any prime ideal, then $\dim \operatorname{Spec} k[x_1, \ldots, x_n]/\mathfrak{p}$ is the transcendence degree of $k[x_1, \ldots, x_n]/\mathfrak{p}$ over k.

Before getting to the proof, I want to discuss some consequences.

2.3. *Corollary.* — $\dim \mathbb{A}^n_k = n$.

We can now confirm that we have named all the primes of $k[x, y]$ where k is algebraically closed. Recall that we have discovered the primes (0), $f(x, y)$ where f is irreducible, and $(x - a, y - b)$ where $a, b \in k$. By the Nullstellensatz, we have found all the closed points, so we have found all the irreducible subsets of dimension 0. As $\mathbb{A}^2_k$ is irreducible, there is only one irreducible subset of dimension 2. So it remains to show that all the irreducible subsets of dimension 1 are of the form $V(f(x, y))$, where f is an irreducible polynomial. Suppose $\mathfrak{p}$ is a prime ideal corresponding to an irreducible subset of dimension 1. Suppose $g \in \mathfrak{p}$ is non-zero. Factor g into irreducibles: $g_1 \cdots g_n \in \mathfrak{p}$. Then as $\mathfrak{p}$ is prime, one of the g_i's, say g_1, lies in $\mathfrak{p}$. Thus $(g_1) \subset \mathfrak{p}$. Now (g_1) is a prime ideal, and hence cuts out an irreducible subset, which contains $V(\mathfrak{p})$. It can't strictly contain $V(\mathfrak{p})$, as its dimension is no bigger than 1, and the dimension of $V(\mathfrak{p})$ is also 1. Thus $V((g_1)) = V(\mathfrak{p})$. But they are both prime ideals, and by the bijection between irreducible closed subsets and prime ideals, we have $\mathfrak{p} = (g_1)$.

Here are two more exercises added to the notes on November 5.

2.4. Exercise: Nullstellensatz from dimension theory. (a) Prove a microscopically stronger version of the weak Nullstellensatz: Suppose $R = k[x_1, \ldots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \ldots, x_n - a_n)$, where $a_i \in k$.
(b) Suppose $R = k[x_1, \ldots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of R has a residue field that is a finite extension of k. [Hint for both: the maximal ideals correspond to dimension 0 points, which correspond to transcendence

degree 0 extensions of k, i.e. finite extensions of k. If k is algebraically closed, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.]

2.5. Important Exercise. Suppose X is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where R is a finitely generated domain over k. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X.

Here is an application that you might reasonably have wondered about before thinking about algebraic geometry. I don't think there is a simple proof, but maybe I'm wrong.

2.6. *Exercise.* Suppose $f(x, y)$ and $g(x, y)$ are two complex polynomials ($f, g \in \mathbb{C}[x, y]$). Suppose f and g have no common factors. Show that the system of equations $f(x, y) = g(x, y) = 0$ has a finite number of solutions.

Let's start to prove the big theorem! If R is a finitely generated domain over k, temporarily define $\dim_{tr} R = \dim_{tr} \operatorname{Spec} R$ to be the transcendence degree of $\operatorname{Frac}(R)$ over k. We wish to show that $\dim_{tr} R = \dim R$. After proving the big theorem, we will discard the temporary notation $\dim_{tr}$.

2.7. *Lemma. — Suppose R is an integral domain over k (not necessarily finitely generated, although that is the case we will care most about), and $\mathfrak{p} \subset R$ a prime. Then $\dim_{tr} R \geq \dim_{tr} R/\mathfrak{p}$, with equality if and only $\mathfrak{p} = (0)$, or $\dim_{tr} R/\mathfrak{p} = \infty$.*

You should have a picture in your mind when you hear this: if you have an irreducible space of finite dimension, then any proper subspace has strictly smaller dimension — certainly believable!

This implies that $\dim R \leq \dim_{tr} R$.

Proof. You can quickly check that if $\mathfrak{p} = (0)$ or $\dim_{tr} R/\mathfrak{p} = \infty$ then we have equality, so we'll assume that $\mathfrak{p} \neq (0)$, and $\dim_{tr} R/\mathfrak{p} = n < \infty$. Choose $x_1, \dots, x_n$ in R such that their residues $\overline{x}_1, \dots, \overline{x}_n$ are algebraically independent. Choose any $y \neq 0$ in $\mathfrak{p}$. Assume for the sake of contradiction that $\dim_{tr} R = n$. Then $y, x_1, \dots, x_n$ cannot be algebraically independent over k, so there is some irreducible polynomial $f(Y, X_1, \dots, X_n) \in k[Y, X_1, \dots, X_n]$ such that $f(y, x_1, \dots, x_n) = 0$ (in R). This irreducible f is not (a multiple of) Y, as otherwise $f(y, x_1, \dots, x_n) = y \neq 0$ in R. Hence f contains monomials that are not multiples of Y, so $F(X_1, \dots, X_n) := f(0, X_1, \dots, X_n) \in k[X_1, \dots, X_n]$ is non-zero. Reducing $f(y, x_1, \dots, x_n) = 0$ modulo $\mathfrak{p}$ gives us

$$F(\overline{x}_1, \dots, \overline{x}_n) = f(0, \overline{x}_1, \dots, \overline{x}_n) = 0 \qquad \text{in } R/\mathfrak{p}$$

contradicting the algebraic independence of $\overline{x}_1, \dots, \overline{x}_n$. $\square$

At the end of the class, I stated the following, which will play off of our lemma to prove the big theorem.

2.8. *Krull's principal ideal theorem (transcendence degree version). — Suppose* R *is a finitely generated domain over* k, $f \in R$, $\mathfrak{p}$ *a minimal prime of* R/f*. Then if* $f \neq 0$, $\dim_{\mathrm{tr}} R/\mathfrak{p} = \dim_{\mathrm{tr}} R - 1$.

This is best understood geometrically:

2.9. *Theorem (geometric interpretation of Krull). — Suppose* $X = \operatorname{Spec} R$ *where* R *is a finitely generated domain over* k, $g \in R$, Z *an irreducible component of* $V(g)$*. Then if* $g \neq 0$, $\dim_{\mathrm{tr}} Z = \dim_{\mathrm{tr}} X - 1$.

In other words, if you have some irreducible space of finite dimension, then any non-zero function on it cuts out a set of *pure* codimension 1.

We'll see how these two geometric statements will quickly combine to prove our big theorem.

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