FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 7

CONTENTS

Last day: The structure sheaf.

Today: $\tilde{M}$, sheaf associated to R-module M; Chinese remainder theorem; germs and value at a point of the structure sheaf; non-affine schemes $\mathbb{A}^2 - (0,0)$, line with doubled origin, $\mathbb{P}^n$.

Another problem set 2 issue, about the pullback sheaf. First, I think I'd like to call it the inverse image sheaf, because I don't want to confuse it with something that I'll also call the pullback. Second, and more importantly, I didn't give the correct definition.

Here is what I should have said (and what is now in the problem set). Define $f^{-1}\mathcal{G}^{\text{pre}}(U) = \lim_{\rightarrow V \supset f(U)} \mathcal{G}(V)$. Then show that this is a *presheaf*. Then the sheafification of this is said to be the *inverse image sheaf* (sometimes called the pullback sheaf) $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\text{pre}})^{\text{sh}}$. Thanks to Kate for pointing out this important patch!

1. PLAYING WITH THE STRUCTURE SHEAF

Here's where we were last day. We defined the *structure sheaf* $\mathcal{O}_{\operatorname{Spec} R}$ on an affine scheme $\operatorname{Spec} R$. We did this by describing it as a *sheaf on the distinguished base*.

An immediate consequence is that we can recover our ring R from the scheme $\operatorname{Spec} R$ by taking global sections, as the entire scheme is $D(1)$:

$$\begin{aligned} \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) &= \Gamma(D(1), \mathcal{O}_{\operatorname{Spec} R}) \quad \text{as } D(1) = \operatorname{Spec} R \\ &= R_1 \quad \text{(i.e. allow 1's in the denominator) by definition} \\ &= R \end{aligned}$$

Another easy consequence is that the restriction of the sheaf $\mathcal{O}_{\operatorname{Spec} R}$ to the distinguished open set $D(f)$ gives us the affine scheme $\operatorname{Spec} R_f$. Thus not only does the set restrict, but also the topology (as we've seen), and the structure sheaf (as this exercise shows).

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1.1. *Important but easy exercise.* Suppose $f \in R$. Show that under the identification of $D(f)$ in $\operatorname{Spec} R$ with $\operatorname{Spec} R_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}) \cong (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$.

The proof of Big Theorem of last time (that the object $\mathcal{O}_{\operatorname{Spec} R}$ defined by $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} R}) = R_f$ forms a sheaf on the distinguished base, and hence a sheaf) immediately generalizes, as the following exercise shows. This exercise will be essential for the definition of a quasicoherent sheaf later on.

1.2. *Important but easy exercise.* Suppose M is an R-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_R R_f$; the restriction map is the "obvious" one. This is a sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules! We get a natural bijection: rings, modules $\leftrightarrow$ Affine schemes, $\tilde{M}$.

Useful fact for later: As a consequence, note that if $(f_1, \ldots, f_r) = R$, we have identified M with a specific submodule of $M_{f_1} \times \cdots \times M_{f_r}$. Even though $M \to M_{f_i}$ may not be an inclusion for any f_i, $M \to M_{f_1} \times \cdots \times M_{f_r}$ *is* an inclusion. We don't care yet, but we'll care about this later, and I'll invoke this fact. (Reason: we'll want to show that if M has some nice property, then M_f does too, which will be easy. We'll also want to show that if $(f_1, \ldots, f_n) = R$, then if M_{f_i} have this property, then M does too.)

1.3. Brief fun: The Chinese Remainder Theorem is a *geometric* fact. I want to show you that the Chinese Remainder theorem is embedded in what we've done, which shouldn't be obvious to you. I'll show this by example. The Chinese Remainder Theorem says that knowing an integer modulo 60 is the same as knowing an integer modulo 3, 4, and 5. Here's how to see this in the language of schemes. What is $\operatorname{Spec} \mathbb{Z}/(60)$? What are the primes of this ring? Answer: those prime ideals containing (60), i.e. those primes dividing 60, i.e. (2), (3), and (5). So here is my picture of the scheme [3 dots]. They are all closed points, as these are all maximal ideals, so the topology is the discrete topology. What are the stalks? You can check that they are $\mathbb{Z}/4$, $\mathbb{Z}/3$, and $\mathbb{Z}/5$. My picture is actually like this (draw a small arrow on (2)): the scheme has nilpotents here ($2^2 \equiv 0 \pmod 4$). So what are global sections on this scheme? They are sections on this open set (2), this other open set (3), and this third open set (5). In other words, we have a natural isomorphism of rings

$$\mathbb{Z}/60 \to \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

On a related note:

1.4. *Exercise.* Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)

This is *always* false for an infinite number of affine schemes:

1.5. *Unimportant exercise.* An infinite disjoint union of (non-empty) affine schemes is not an affine scheme. (One-word hint: quasicompactness.)

1.6. Stalks of this sheaf: germs, and values at a point. Like every sheaf, the structure sheaf has stalks, and we shouldn't be surprised if they are interesting from an algebraic point of view. In fact, we have seen them before.

1.7. *Exercise.* Show that the stalk of $\mathcal{O}_{\operatorname{Spec} R}$ at the point $[\mathfrak{p}]$ is the ring $R_{\mathfrak{p}}$. (Hint: use distinguished open sets in the direct limit you use to define the stalk. In the course of doing this, you'll discover a useful principle. In the concrete definition of stalk, the elements were sections of the sheaf over *some* open set containing our point, and two sections over different open sets were considered the same if they agreed on some smaller open set. In fact, you can just consider elements of your base when doing this. This is called a *cofinal system* in the *directed set*.) This is yet another reason to like the notion of a sheaf on a base.

The *residue field of a scheme at a point* is the local ring modulo its maximal ideal.

Essentially the same argument will show that the stalk of the sheaf $\tilde{M}$ at $[\mathfrak{p}]$ is $M_{\mathfrak{p}}$.

So now we can make precise some of our intuition. Suppose $[\mathfrak{p}]$ is a point in some open set U of $\operatorname{Spec} R$. For example, say $R = k[x, y]$, $\mathfrak{p} = (x)$ (draw picture), and $U = \mathbb{A}^2 - (0, 0)$. (First, make sure you see that this *is* an open set! $(0, 0) = V((x, y))$ is indeed closed. Make sure you see that $[\mathfrak{p}]$ indeed is in U.)

- Then a function on U, i.e. a section of $\mathcal{O}_{\operatorname{Spec} R}$ over U, has a *germ near* $[\mathfrak{p}]$, which is an element of $R_{\mathfrak{p}}$. Note that this is a local ring, with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. In our example, consider the function $(3x^4 + x^2 + xy + y^2)/(3x^2 + xy + y^2 + 1)$, which is defined on the open set $D(3x^2 + xy + y^2 + 1)$. Because there are no factors of x in the denominator, it is indeed in $R_{\mathfrak{p}}$.
- A germ has a *value* at $[\mathfrak{p}]$, which is an element of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. (This is isomorphic to $\operatorname{Frac}(R/\mathfrak{p})$, the fraction field of the quotient domain.) So the value of a function at a point always takes values in a field. In our example, to see the value of our germ at $x = 0$, we simply set $x = 0$! So we get the value $y^2/(y^2 + 1)$, which is certainly in $\operatorname{Frac} k[y]$.
- We say that the germ *vanishes* at $\mathfrak{p}$ if the value is zero. In our example, the germ doesn't vanish at $\mathfrak{p}$.

If anything makes you nervous, you should make up an example to assuage your nervousness. (Example: $27/4$ is a regular function on $\operatorname{Spec} \mathbb{Z} - \{(2), (7)\}$. What is its value at (5)? Answer: $2/(-1) \equiv -2 \pmod 5$. What is its value at (0)? Answer: $27/4$. Where does it vanish? At (3).)

1.8. An extended example. I now want to work through an example with you, to show that this distinguished base is indeed something that you can work with. Let $R = k[x, y]$, so $\operatorname{Spec} R = \mathbb{A}^2_k$. If you want, you can let k be $\mathbb{C}$, but that won't be relevant. Let's work out the space of functions on the open set $U = \mathbb{A}^2 - (0, 0)$.

It is a non-obvious fact that you can't cut out this set with a single equation, so this isn't a distinguished open set. We'll see why fairly soon. But in any case, even if we're not sure if this is a distinguished open set, we can describe it as the union of two things which *are* distinguished open sets. If I throw out the x axis, i.e. the set $y = 0$, I get the distinguished open set $D(y)$. If I throw out the y axis, i.e. $x = 0$, I get the distinguished open set $D(x)$. So $U = D(x) \cup D(y)$. (Remark: $U = \mathbb{A}^2 - V(x, y)$ and $U = D(x) \cup D(y)$. Coincidence? I think not!) We will find the functions on U by gluing together functions on $D(x)$ and $D(y)$.

What are the functions on $D(x)$? They are, by definition, $R_x = k[x, y, 1/x]$. In other words, they are things like this: $3x^2 + xy + 3y/x + 14/x^4$. What are the functions on $D(y)$? They are, by definition, $R_y = k[x, y, 1/y]$. Note that $R \hookrightarrow R_x, R_y$. This is because x and y are not zero-divisors. (In fact, R is an integral domain — it has no zero-divisors, besides 0 — so localization is always an inclusion.) So we are looking for functions on $D(x)$ and $D(y)$ that agree on $D(x) \cap D(y) = D(xy)$, i.e. they are just the same function. Well, which things of this first form are also of the second form? Just old-fashioned polynomials —

$$\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) \equiv k[x, y]. \tag{1}$$

In other words, we get no extra functions by throwing out this point. Notice how easy that was to calculate!

This is interesting: any function on $\mathbb{A}^2 - (0, 0)$ extends over all of $\mathbb{A}^2$. (Aside: This is an analog of Hartogs' theorem in complex geometry: you can extend a holomorphic function defined on the complement of a set of codimension at least two on a complex manifold over the missing set. This will work more generally in the algebraic setting: you can extend over points in codimension at least 2 not only if they are smooth, but also if they are mildly singular — what we will call *normal*.)

We can now see that this is not an affine scheme. Here's why: otherwise, if $(U, \mathcal{O}_{\mathbb{A}^2}|_U) = (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$, then we can recover S by taking global sections:

$$S = \Gamma(U, \mathcal{O}_{\mathbb{A}^2}|_U),$$

which we have already identified in (1) as $k[x, y]$. So if U is affine, then $U = \mathbb{A}^2_k$. But we get more: we can recover the points of $\operatorname{Spec} S$ by taking the primes of S. In particular, the prime ideal (x, y) of S should cut out a point of $\operatorname{Spec} S$. But on U, $V(x) \cap V(y) = \emptyset$. Conclusion: U is *not* an affine scheme. (If you are ever looking for a counterexample to something, and you are expecting one involving a non-affine scheme, keep this example in mind!)

It is however a scheme.

Again, let me repeat the *definition of a* **scheme**. It is a topological space X, along with a sheaf of rings $\mathcal{O}_X$, such that any point $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme (i.e. we have a homeomorphism of U with some $\operatorname{Spec} R$, say $f : U \to \operatorname{Spec} R$, and an isomorphism $\mathcal{O}_X|_U \cong \mathcal{O}_R$, where the two spaces are identified). The scheme can be denoted $(X, \mathcal{O}_X)$, although it is often denoted X, with the structure sheaf implicit.

I stated earlier in the notes, Exercise 1.1 (and at roughly at this point in the class): If we take the underlying subset of $D(f)$ with the restriction of the sheaf $\mathcal{O}_{\operatorname{Spec} R}$, we obtain the scheme $\operatorname{Spec} R_f$.

If X is a scheme, and U is *any* open subset, then $(U, \mathcal{O}_X|_U)$ is also a scheme. **Exercise.** Prove this. $(U, \mathcal{O}_X|_U)$ is called an *open subscheme of* U. If U is also an affine scheme, we often say U is an *affine open subset*, or an *affine open subscheme*, or sometimes informally just an *affine open*. For an example, $D(f)$ is an affine open subscheme of $\operatorname{Spec} R$.

1.9. *Exercise.* Show that if X is a scheme, then the affine open sets form a base for the Zariski topology. (Warning: they don't form a nice base, as we'll see in Exercise 1.11 below. However, in "most nice situations" this will be true, as we will later see, when we define the analogue of "Hausdorffness", called separatedness.)

You've already seen two examples of non-affine schemes: an infinite disjoint union of non-empty schemes, and $\mathbb{A}^2 - (0,0)$. I want to give you two more important examples. They are important because they are the first examples of fundamental behavior, the first pathological, and the second central.

First, I need to tell you how to glue two schemes together. Suppose you have two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, and open subsets $U \subset X$ and $V \subset Y$, along with a homeomorphism $U \cong V$, and an isomorphism of structure sheaves $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$. Then we can glue these together to get a single scheme. Reason: let W be X and Y glued together along the isomorphism $U \cong V$. Then problem 9 on the second problem set shows that the structure sheaves can be glued together to get a sheaf of rings. Note that this is indeed a scheme: any point has a neighborhood that is an affine scheme. (Do you see why?)

So I'll give you two non-affine schemes. In both cases, I will glue together two copies of the affine line $\mathbb{A}^1_k$. Again, if it makes you feel better, let $k = \mathbb{C}$, but it really doesn't matter; this is the last time I'll say this.

Let $X = \operatorname{Spec} k[t]$, and $Y = \operatorname{Spec} k[u]$. Let $U = D(t) = \operatorname{Spec} k[t, 1/t] \subset X$ and $V = D(u) = \operatorname{Spec} k[u, 1/u] \subset Y$.

We will get example 1 by gluing X and Y together along U and V. We will get example 2 by gluing X and Y together along U and V.

Example 1: the affine line with the doubled origin. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow u$. Let the resulting scheme be X. The picture looks like this [line with doubled origin]. This is called the *affine line with doubled origin*.

As the picture suggests, intuitively this is an analogue of a failure of Hausdorffness. $\mathbb{A}^1$ itself is not Hausdorff, so we can't say that it is a failure of Hausdorffness. We see this as weird and bad, so we'll want to make up some definition that will prevent this from happening. This will be the notion of *separatedness*. This will answer other of our prayers as well. For example, on a separated scheme, the "affine base of the Zariski topology" is nice — the intersection of two affine open sets will be affine.

1.10. *Exercise.* Show that X is not affine. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - (0,0)$.

1.11. *Exercise.* Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You'll have defined the *affine plane with doubled origin*. Use this example to show that the affine base of the topology isn't a nice base, by describing two affine open sets whose intersection is not affine.

Example 2: the projective line. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow 1/u$. The picture looks like this [draw it]. Call the resulting scheme the *projective line over the field* k, $\mathbb{P}^1_k$.

Notice how the points glue. Let me assume that k is algebraically closed for convenience. (You can think about how this changes otherwise.) On the first affine line, we have the closed (= "old-fashioned") points $(t - a)$, which we think of as "a on the t-line", and we have the generic point. On the second affine line, we have closed points that are "b on the u-line", and the generic point. Then a on the t-line is glued to $1/a$ on the u-line (if $a \neq 0$ of course), and the generic point is glued to the generic point (the ideal (0) of $k[t]$ becomes the ideal (0) of $k[t, 1/t]$ upon localization, and the ideal (0) of $k[u]$ becomes the ideal (0) of $k[u, 1/u]$. And (0) in $k[t, 1/t]$ is (0) in $k[u, 1/u]$ under the isomorphism $t \leftrightarrow 1/u$).

We can interpret the closed ("old-fashioned") points of $\mathbb{P}^1$ in the following way, which may make this sound closer to the way you have seen projective space defined earlier. The points are of the form $[a; b]$, where a and b are not both zero, and $[a; b]$ is identified with $[ac; bc]$ where $c \in k^*$. Then if $b \neq 0$, this is identified with a/b on the t-line, and if $a \neq 0$, this is identified with b/a on the u-line.

1.12. *Exercise.* Show that $\mathbb{P}^1_k$ is not affine. Hint: calculate the ring of global sections.

This one I will do for you.

The global sections correspond to sections over X and sections over Y that agree on the overlap. A section on X is a polynomial $f(t)$. A section on Y is a polynomial $g(u)$. If I restrict $f(t)$ to the overlap, I get something I can still call $f(t)$; and ditto for $g(u)$. Now we want them to be equal: $f(t) = g(1/t)$. How many polynomials in t are at the same time polynomials in $1/t$? Not very many! Answer: only the constants k. Thus $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. If $\mathbb{P}^1$ were affine, then it would be $\operatorname{Spec} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Spec} k$, i.e. one point. But it isn't — it has lots of points.

Note that we have proved an analog of a theorem: the only holomorphic functions on $\mathbb{C}\mathbb{P}^1$ are the constants!

1.13. *Important exercise.* Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n + 1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

Exercise. The closed points of $\mathbb{P}^n_k$ (if k is algebraically closed) may be interpreted in the same way as we interpreted the points of $\mathbb{P}^1_k$. The points are of the form $[a_0; \ldots; a_n]$, where the a_i are not all zero, and $[a_0; \ldots; a_n]$ is identified with $[ca_0; \ldots; ca_n]$ where $c \in k^*$.

We will later give another definition of projective space. Your definition will be handy for computing things. But there is something unnatural about it — projective space is highly symmetric, and that isn't clear from your point of view.

Note that your definition will give a definition of $\mathbb{P}^n_R$ for any *ring* R. This will be useful later.

1.14. *Example.* Here is an example of a function on an open subset of a scheme that is a bit surprising. On $X = \operatorname{Spec} k[w, x, y, z]/(wx - yz)$, consider the open subset $D(y) \cup D(w)$. Show that the function x/y on $D(y)$ agrees with z/w on $D(w)$ on their overlap $D(y) \cap D(w)$. Hence they glue together to give a section. Justin points out that you may have seen this before when thinking about analytic continuation in complex geometry — we have a "holomorphic" function the description x/y on an open set, and this description breaks down elsewhere, but you can still "analytically continue" it by giving the function a different definition.

Follow-up for curious experts: This function has no "single description" as a well-defined expression in terms of w, x, y, z! there is lots of interesting geometry here. Here is a glimpse, in terms of words we have not yet defined. $\operatorname{Spec} k[w, x, y, z]$ is $\mathbb{A}^4$, and is, not surprisingly, 4-dimensional. We are looking at the set X, which is a hypersurface, and is 3-dimensional. It is a cone over a smooth quadric surface in $\mathbb{P}^3$ [show them hyperboloid of one sheet, and point out the two rulings]. $D(y)$ is X minus some hypersurface, so we are throwing away a codimension 1 locus. $D(z)$ involves throwing another codimension 1 locus. You might think that their intersection is then codimension 2, and that maybe failure of extending this weird function to a global polynomial comes because of a failure of our Hartogs'-type theorem, which will be a failure of normality. But that's not true — $V(y) \cap V(z)$ is in fact codimension 1 — so no Hartogs-type theorem holds. Here is what is actually going on. $V(y)$ involves throwing away the (cone over the) union of two lines l and m_1, one in each "ruling" of the surface, and $V(z)$ also involves throwing away the (cone over the) union of two lines l and m_2. The intersection is the (cone over the) line l, which is a codimension 1 set. Neat fact: despite being "pure codimension 1", it is not cut out even set-theoretically by a single equation. (It is hard to get an example of this behavior. This example is the simplest example I know.) This means that any expression $f(w, x, y, z)/g(w, x, y, z)$ for our function cannot correctly describe our function on $D(y) \cup D(z)$ — at some point of $D(y) \cup D(z)$ it must be $0/0$. Here's why. Our function can't be defined on $V(y) \cap V(z)$, so g must vanish here. But then g can't vanish just on the cone over l — it must vanish elsewhere too. (For the experts among the experts: here is why the cone over l is not cut out set-theoretically by a single equation. If $l = V(f)$, then $D(f)$ is affine. Let l' be another line in the same ruling as l, and let $C(l)$ (resp. l' be the cone over l (resp. l'). Then $C(l')$ can be given the structure of a closed subscheme of $\operatorname{Spec} k[w, x, y, z]$, and can be given the structure of $\mathbb{A}^2$. Then $C(l') \cap V(f)$ is a closed subscheme of $D(f)$. Any closed subscheme of an affine scheme is affine. But $l \cap l' = \emptyset$, so the cone over l intersects the cone over l' is a point, so $C(l') \cap V(f)$ is $\mathbb{A}^2$ minus a point, which we've seen is not affine, contradiction.)

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