FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 6

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CONTENTS

Last day: $\operatorname{Spec} R$**: the set, and the topology**

Today: The structure sheaf, and schemes in general.

Announcements: on problem set 2, there was a serious typo in # 10. $\operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U))$ should read $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. The notation is new, but will likely be clear to you after you think about it a little. If $\mathcal{F}$ is a sheaf on X, and U is an open subset, then we can define the sheaf $\mathcal{F}|_U$ on U in the obvious way. This is sometimes called *the restriction of the sheaf* $\mathcal{F}$ *to the open set* U (not to be confused with restriction maps!). This homomorphism $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of all *sheaf homomorphisms* from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. The revised version is posted on the website.

Also, the final problem set this quarter will be due no later than Monday, December 12, the Monday after the last class.

1. RECAP OF LAST DAY, AND FURTHER DISCUSSION

Last day, we saw lots of examples of the underlying sets of affine schemes, which correspond to primes in a ring. In this dictionary, "an element r of the ring lying in a prime ideal $\mathfrak{p}$" translates to "an element r of the ring vanishing at the point $[\mathfrak{p}]$, and I will use these phrases interchangeably.

There was some language I was using informally, and I've decided to make it more formal: elements $r \in R$ will officially be called "global functions", and their value at the point $[\mathfrak{p}]$ will be $r \pmod{\mathfrak{p}}$. This language will be "justified" by the end of today.

I then defined the Zariski topology. The closed subsets were just those points where some set of ring elements all vanish.

Date: Wednesday, October 19, 2005. Minor update January 30, 2007.

As a reminder, here are the key words that we learned about topological spaces: irreducible; generic point; closed points (points $\mathfrak{p}$ such that $\overline{\{\mathfrak{p}\}} = \{\mathfrak{p}\}$; did I forget to say this last time?); specialization/generization; quasicompact. All of these words can be used on any topological spaces, but they tend to be boring (or highly improbable) on spaces that you knew and loved before.

On $\operatorname{Spec} R$, closed points correspond to maximal ideals of R. Also, I described a bijection between closed subsets and radical ideals. The two maps of this bijection use the "vanishing set" function V and the "ideal of functions vanishing" function I. I also described a bijection between points and closed subsets; one direction involved taking closures, and the other involved taking generic points. Some of this was left to you in the form of **exercises**.

As an example, consider the prime (or point) $(y - x^2)$ in $k[x, y]$ (or $\operatorname{Spec} k[x, y]$). What is its closure? We look at all functions vanishing at this point, and see at what other points they all vanish. In other words, we look for all prime ideals containing all elements of this one. In other words, we look at all prime ideals containing this one. Picture: we get all the closed points on the parabola. We get the closed set corresponding to this point. (Caveat: I haven't proved that I've described all the primes in $k[x, y]$.)

In class, I spontaneously showed you that the Zariski closure of the countable set (n, n^2) in $\mathbb{A}^2_{\mathbb{C}}$ was the parabola. The Zariski closure of a finite set of points will just be itself: a finite union of closed sets is closed.

Last day I showed that the Zariski topology behaves well with respect to taking quotients, and localizing. I said a little more today.

About taking quotients: suppose you have a ring R, and an ideal I. Then there is a bijection of the points of $\operatorname{Spec} R/I$ with the points of $V(I)$ in $\operatorname{Spec} R$. (Just unwind the algebraic definition! Both correspond to primes $\mathfrak{p}$ of R containing I.) My comments of last day showed that this is in fact a homeomorphism: $\operatorname{Spec} R/I$ may be identified with the closed subset of $\operatorname{Spec} R$ as a topological space: the subspace topology induced from $\operatorname{Spec} R$ is indeed the topology of $\operatorname{Spec} R$. The reason was not sophisticated: there is a natural correspondence of closed subsets.

About localizing: this is quite a general procedure, so in general you can't say much besides the fact that $\operatorname{Spec} S^{-1}R$ is naturally a subset of $\operatorname{Spec} R$, with the induced topology. Last day I discussed the important case where $S = R - \mathfrak{p}$, the complement of a prime ideal, so then $S^{-1}R = R_{\mathfrak{p}}$.

But there is a second important example of localization, when $S = \{1, f, f^2, \dots\}$ for some $f \in R$. In this case we get the ring denoted R_f. In this case, $\operatorname{Spec} R_f$ is $D(f)$, again just by unwinding the definitions: both consist of the primes not containing f (= the points where f doesn't vanish). The Zariski topology on $D(f)$ agrees with the Zariski topology on $\operatorname{Spec} R_f$.

Here is an **exercise** from last day. Show that $(f_1, \dots, f_n) = (1)$ if and only if $\cup D(f_i) = \operatorname{Spec} R$. I want to do it for you, to show you how it can be interpreted simultaneously in both algebra and geometry. Here is one direction. Suppose $[\mathfrak{p}] \notin \cup D(f_i)$. You can unwind this to get an algebraic statement. I think of it as follows. All of the f_i vanish at $[\mathfrak{p}]$, i.e. all $f_i \in \mathfrak{p}$, so then $(f_1, \dots) \subset \mathfrak{p}$ and hence this ideal can't be all of R. Conversely, consider the ideal $(f_1, \dots)$. If it isn't R, then it is contained in a maximal ideal. (For logic-lovers: we're using the axiom of choice, which I said I'd assume at the very start of this class.) But then there is some prime ideal containing all the f_i. Translation: $[\mathfrak{m}] \notin D(f_i)$ for any i. (As an added bonus: this argument shows that if $\operatorname{Spec} R$ is an infinite union $\cup_{i \in I} D(f_i)$, then in fact it is a union of a finite number of these. This is one way of proving quasicompactness.)

Important comment: This machinery will let us bring our geometric intuition to algebra. There is one point where your intuition will be false, and I want to tell you now, so you can adjust your intuition appropriately. Suppose we have a function (ring element) vanishing at all points. Is it zero? Not necessarily! Translation: is intersection of all prime ideals necessarily just 0? No: $k[\epsilon]/\epsilon^2$ is a good example, as $\epsilon \neq 0$, but $\epsilon^2 = 0$. This is called the *ring of dual numbers* (over the field k). Any function whose power is zero certainly lies in the intersection of all prime ideals. And the converse is true (algebraic fact): the intersection of all the prime ideals consists of functions for which some power is zero, otherwise known as the nilradical $\mathfrak{N}$. (Ring elements that have a power that is 0 are called *nilpotents*.) Summary: "functions on affine schemes" will not be determined by their values at points. (For example: $\operatorname{Spec} k[\epsilon]/\epsilon^2$ has one point. $3 + 4\epsilon$ has value 3 at that point, but the function isn't 3.) In particular, any function vanishing at all points might not be zero, but some power of it is zero. This takes some getting used to.

1.1. *Easy fun unimportant exercise.* Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x, \epsilon]/\epsilon^2$. What then is $f(x + \epsilon)$? (Do a couple of examples, and you will see the pattern. For example, if $f(x) = 3x^3 + 2x$, we get $f(x + \epsilon) = (3x^3 + 2x) + \epsilon(9x^2 + 2)$. Prove the pattern!) Useful tip: the dual numbers are a good source of (counter)examples, being the "smallest ring with nilpotents". They will also end up being important in defining differential information.

Here is one more (important!) algebraic fact: suppose $D(f) \subset D(g)$. Then $f^n \in (g)$ for some n. I'm going to let you prove this (**Exercise** from last day), but I want to tell you how I think of it geometrically. Draw a picture of $\operatorname{Spec} R$. Draw the closed subset $V(g) = \operatorname{Spec} R/(g)$. That's where g vanishes, and the complement is $D(g)$, where g doesn't vanish. Then f is a function on this closed subset, and it vanishes at all points of the closed subset. (Translation: Consider f as an element of the ring $R/(g)$.) Now any function vanishing at every point of Spec a ring must have some power which is 0. Translation: there is some n such that $f^n = 0$ in $R/(g)$, i.e. $f^n \equiv 0 \pmod g$ in R, i.e. $f^n \in (g)$.

2. THE FINAL INGREDIENT IN THE DEFINITION OF AFFINE SCHEMES: THE STRUCTURE SHEAF

The final ingredient in the definition of an affine scheme is the **structure sheaf** $\mathcal{O}_{\operatorname{Spec} R}$, which we think of as the "sheaf of algebraic functions". These functions will have values at points, but won't be determined by their values at points. Like all sheaves, they will indeed be determined by their germs.

It suffices to describe it as a sheaf on the nice base of distinguished open sets. We define the sections on the base by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \tag{1}$$

We define the restriction maps $\operatorname{res}_{D(g),D(f)}$ as follows. If $D(f) \subset D(g)$, then we have shown that $f^n \in (g)$, i.e. we can write $f^n = ag$. There is a natural map $R_g \to R_f$ given by $r/g^m \mapsto (ra^m)/(f^{mn})$, and we define

$$\operatorname{res}_{D(g),D(f)} : \mathcal{O}_{\operatorname{Spec} R}(D(g)) \to \mathcal{O}_{\operatorname{Spec} R}(D(f))$$

to be this map.

2.1. *Exercise.* (a) Verify that (1) is well-defined, i.e. if $D(f) = D(f')$ then R_f is canonically isomorphic to $R_{f'}$. (b) Show that $\operatorname{res}_{D(g),D(f)}$ is well-defined, i.e. that it is independent of the choice of a and n, and if $D(f) = D(f')$ and $D(g) = D(g')$, then

$$\begin{array}{ccc} R_g & \xrightarrow{\operatorname{res}_{D(g),D(f)}} & R_f \\ \downarrow \sim & & \downarrow \sim \\ R_{g'} & \xrightarrow{\operatorname{res}_{D(g'),D(f')}} & R_{f'} \end{array}$$

commutes.

We now come to the big theorem of today.

2.2. *Theorem. — The data just described gives a sheaf on the (nice) distinguished base, and hence determines a sheaf on the topological space* $\operatorname{Spec} R$.

This sheaf is called the **structure sheaf**, and will be denoted $\mathcal{O}_{\operatorname{Spec} R}$, or sometimes $\mathcal{O}$ if the scheme in question is clear from the context. Such a topological space, with sheaf, will be called an **affine scheme**. The notation $\operatorname{Spec} R$ will hereafter be a topological space, with a structure sheaf.

Proof. Clearly this is a presheaf on the base: if $D(f) \subset D(g) \subset D(h)$ then the following diagram commutes:

$$\begin{array}{ccc} R_h & \xrightarrow{\operatorname{res}_{D(h),D(g)}} & R_g \\ & \operatorname{res}_{D(h),D(f)} \searrow \quad \swarrow \operatorname{res}_{D(g),D(f)} & \\ & R_f. & \end{array} \tag{2}$$

You can check this directly. Here is a trick which helps (and may help you with Exercise 2.1 above). As $D(g) \subset D(h)$, $D(gh) = D(g)$. (Translation: The locus where g doesn't vanish is a subset of where h doesn't vanish, so the locus where gh doesn't vanish is the same as the locus where g doesn't vanish.) So we can replace R_g by R_{gh}, and R_f by R_{fgh}, in (2). The restriction maps are $\operatorname{res}_{D(h),D(gh)} : a/h \mapsto ag/gh$, $\operatorname{res}_{D(gh),D(fgh)} : b/gh \mapsto bf/fgh$, and $\operatorname{res}_{D(h),D(fgh)} : a/h \to afg/fgh$, so they clearly commute as desired.

We next check identity on the base. We deal with the case of a cover of the entire space R, and let the reader verify that essentially the same argument holds for a cover of some R_f. Suppose that $\operatorname{Spec} R = \cup_{i \in A} D(f_i)$ where i runs over some index set I. By quasicompactness, there is some finite subset of I, which we name $\{1, \dots, n\}$, such that $\operatorname{Spec} R = \cup_{i=1}^n D(f_i)$, i.e. $(f_1, \dots, f_n) = R$. (Now you see why we like quasicompactness!) Suppose we are given $s \in R$ such that $\operatorname{res}_{\operatorname{Spec} R, D(f_i)} s = 0$ in R_{f_i} for all i. Hence there is some m such that for each $i \in \{1, \dots, n\}$, $f_i^m s = 0$. (Reminder: $R \to R_f$. What goes to 0? Precisely things killed by some power of f.) Now $(f_1^m, \dots, f_n^m) = R$ (do you know why?), so there are $r_i \in R$ with $\sum_{i=1}^n r_i f_i^m = 1$ in R, from which

$$s = \left(\sum r_i f_i^m \right) s = 0.$$

Thus we have checked the "base identity" axiom for $\operatorname{Spec} R$.

Remark. Serre has described this as a "partition of unity" argument, and if you look at it in the right way, his insight is very enlightening.

2.3. *Exercise.* Make the tiny changes to the above argument to show base identity for any distinguished open $D(f)$.

We next show base gluability. As with base identity, we deal with the case where we wish to glue sections to produce a section over $\operatorname{Spec} R$. As before, we leave the general case where we wish to glue sections to produce a section over $D(f)$ to the reader (Exercise 2.4).

Suppose $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$, where I is a index set (possibly horribly uncountably infinite). Suppose we are given

$$\frac{a_i}{f_i^{l_i}} \in R_{f_i} \quad (i \in I)$$

such that for all $i, j \in I$, there is some $m_{ij} \geq l_i, l_j$ with

$$(f_i f_j)^{m_{ij}} \left(f_j^{l_j} a_i - f_i^{l_i} a_j \right) = 0 \tag{3}$$

in R. We wish to show that there is some $r \in R$ such that $r = a_i / f_i^{l_i}$ in R_{f_i} for all $i \in I$.

Choose a finite subset $\{1, \dots, n\} \subset I$ with $(f_1, \dots, f_n) = R$.

To save ourself some notation, we may take the l_i to all be 1, by replacing f_i with $f_i^{l_i}$ (as $D(f_i) = D(f_i^{l_i})$). We may take m_{ij} ($1 \leq i, j \leq n$) to be the same, say m — take $m = \max m_{ij}$.) The only reason to do this is to have fewer variables.

Let $a_i' = a_i f_i^m$. Then $a_i / f_i = a_i' / f_i^{m+1}$, and (3) becomes

$$f_j^{m+1} a_i' - f_i^{m+1} a_j' = 0 \tag{4}$$

As $(f_1, \dots, f_n) = R$, we have $(f_1^{m+1}, \dots, f_n^{m+1}) = R$, from which $1 = \sum b_i f_i^{m+1}$ for some $b_i \in R$. Define

$$r = b_1 a_1' + \cdots + b_n a_n'.$$

This will be the r that we seek. For each $i \in \{1, \ldots, n\}$, we will show that $r - a_i'/f_i^{m+1} = 0$ in D_{f_i}. Indeed,

$$
\begin{aligned}
rf_i^{m+1} - a_i' &= \sum_{j=1}^{n} a_j' b_j f_i^{m+1} - \sum_{j=1}^{n} a_i' b_j f_j^{m+1} \\
&= \sum_{j \neq i} b_j (a_j' f_i^{m+1} - a_i' f_j^{m+1}) \\
&= 0 \quad \text{(by (4))}
\end{aligned}
$$

So are we done? No! We are supposed to have something that restricts to $a_i/f_i^{l_i}$ for *all* $i \in I$, not just $i = 1, \ldots, n$. But a short trick takes care of this. We now show that for any $\alpha \in I - \{1, \ldots, n\}$, $r = a_\alpha/f_\alpha^{l_\alpha}$ in R_{f_α}. Repeat the entire process above with $\{1, \ldots, n, \alpha\}$ in place of $\{1, \ldots, n\}$, to obtain $r' \in R$ which restricts to $a_i/f_i^{l_i}$ for $i \in \{1, \ldots, n, \alpha\}$. Then by base identity, $r' = r$. Hence r restricts to $a_\alpha/f_\alpha^{l_\alpha}$ as desired.

2.4. *Exercise.* Alter this argument appropriately to show base gluability for any distinguished open $D(f)$.

□

So now you know what an affine scheme is!

We can even define a scheme in general: it is a topological space X, along with a structure sheaf $\mathcal{O}_X$, that locally looks like an affine scheme: for any $x \in X$, there is an open neighborhood U of x such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

On Friday, I'll discuss some of the ramifications of this definition. In particular, you'll see that stalks of this sheaf are something familiar, and I'll show you that constructing the sheaf by looking at this nice distinguished base isn't just a kluge, it's something very natural — we'll do this by finding sections of $\mathcal{O}_{\mathbb{A}^2}$ over the open set $\mathbb{A}^2 - (0, 0)$.

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