FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 5

CONTENTS

Last day: Understanding sheaves via stalks, and via "nice base" of topology. $\operatorname{Spec} R$**: the set.**

Today: $\operatorname{Spec} R$**: the set, and the Zariski topology.**

Here is a reminder of where we are going. Affine schemes $\operatorname{Spec} R$ will be defined as a topological space with a sheaf of rings, that we will refer to as the sheaf of functions, called the structure sheaf. A scheme in general will be such a thing (a topological space with a sheaf of rings) that locally looks like $\operatorname{Spec} R$'s. Last day we defined the set: it is the set of primes of R.

We're in the process of doing lots of examples. In the course of doing these examples, we are saying things that we aren't allowed to say yet, because we're using words that we haven't defined. We're doing this because it will motivate where we're going.

We discussed the example $R = k[x]$ where k is algebraically closed (notation: $\mathbb{A}^1_k = \operatorname{Spec} k[x]$). This has old-fashioned points $(x - a)$ corresponding to $a \in k$. (Such a point is often just called a, rather than $(x-a)$.) But we have a new point, (0). (Notational caution: $0 \neq (0)$.) This is a "smooth irreducible curve". (We don't know what any of these words mean!)

We then discussed $R = \mathbb{Z}$. This has points (p) where p is an old-fashioned prime, and the point (0). This is also a smooth irreducible curve (whatever that means).

We discussed the case $R = k[x]$ where k is not necessarily algebraically closed, in particular $k = \mathbb{R}$. The maximal ideals here correspond to unions of Galois conjugates of points in $\mathbb{A}^1_{\mathbb{C}}$.

Example 6 for more arithmetic people: $\mathbb{F}_p[x]$. As in the previous examples, this is a Unique Factorization Domain, so we can figure out its primes in a hands-on way. The points are (0), and irreducible polynomials, which come in any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}}_p$. You should think about this, even if you are

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a geometric person — there is some arithmetic intuition that later turns into geometric intuition.

Example 7. $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. (This discussion will apply with $\mathbb{C}$ replaced by *any* algebraically closed field.) Sadly, $\mathbb{C}[x, y]$ is not a Principal Ideal Domain: (x, y) is not a principal ideal. We can quickly name *some* prime ideals. One is (0), which has the same flavor as the (0) ideals in the previous examples. $(x - 2, y - 3)$ is prime, because $\mathbb{C}[x, y]/(x-2, y-3) \cong \mathbb{C}$, where this isomorphism is via $f(x, y) \mapsto f(2, 3)$. More generally, $(x - a, y - b)$ is prime for any $(a, b) \in \mathbb{C}^2$. Also, if $f(x, y)$ is an irreducible polynomial (e.g. $y - x^2$ or $y^2 - x^3$) then $(f(x, y))$ is prime. We will later prove that we have identified all these primes. Here is a picture: the "maximal primes" correspond to the old-fashioned points in $\mathbb{C}^2$ (I drew it). (0) somehow lives behind all of these points (I drew it). $(y - x^2)$ somehow is associated to this parabola (I drew it). Etc. You can see from this picture that we already want to think about "dimension". The primes $(x - a, y - b)$ are somehow of dimension 0, the primes $(f(x, y))$ are of dimension 1, and (0) is somehow of dimension 2. I won't define dimension today, so every time I say it, you should imagine that I am waving my hands wildly.

(This paragraph will not be comprehensible in the notes.) Let's try to picture this. Where is the prime $(y - x^2)$? Well, is it in the plane? Yes. Is it at $(2, 4)$? No. Is it in the set cut out by $y - x^2$? Yes. Is it in the set cut out by $(y^2 - x^3)$? No. Is it in the set cut out by $xy(y - x^2)$? Yes.

Note: maximal ideals correspond to "smallest" points. Smaller ideals correspond to "bigger" points. "One prime ideal contains another" means that the points "have the opposite containment." All of this will be made precise once we have a topology. This order-reversal can be a little confusing (and will remain so even once we have made the notions precise).

Example: $\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$. (More generally, $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$, and even $A^n_R = \operatorname{Spec} R[x_1, \ldots, x_n]$ where R is an arbitrary ring.)

For concreteness, let's consider $n = 3$. We now have an interesting question in algebra: What are the prime ideals of $\mathbb{C}[x, y, z]$? We have $(x-a, y-b, z-c)$. This is a maximal ideal, with residue field $\mathbb{C}$; we think of these as "0-dimensional points". Have we discovered all the maximal ideals? The answer is yes, by Hilbert's Nullstellensatz, which is covered in Math 210.

Hilbert's Nullstellensatz, Version 1. (This is sometimes called the "weak version" of the Nullstellensatz.) Suppose $R = k[x_1, \ldots, x_n]$, where k is an algebraically closed field. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \ldots, x_n - a_n)$, where $a_i \in k$.

There are other prime ideals too. We have (0), which is a "3-dimensional point". We have $(f(x, y, z))$, where f is irreducible. To this we associate the hypersurface $f = 0$, so this is "2-dimensional" in nature. Do we have them all? No! One clue: we're missing dimension 1 things. Here is a "one-dimensional" prime ideal: (x, y). (Picture: it is the z-axis, which is cut out by $x = y = 0$.) How do we check that this is prime? The easiest way is to check that the quotient is an integral domain, and indeed $\mathbb{C}[x, y, z]/(x, y) \cong \mathbb{C}[z]$ is an integral domain (and visibly the functions on the z-axis). There are lots of one-dimensional primes, and it is not possible to classify them in a reasonable way. It will turn out that they correspond to things that we think of as "irreducible" curves: the natural answer to this algebraic question is geometric.

0.1. We now come to two more general flavors of affine schemes that will be useful in the future. There are two nice ways of producing new rings from a ring R. One is by taking the quotient by an ideal I. The other is by localizing at a multiplicative set. We'll see how Spec behaves with respect to these operations. In both cases, the new ring has a Spec that is a subset of the Spec of the old ring.

Important example: $\operatorname{Spec} R/I$ *in terms of* $\operatorname{Spec} R$. As a motivating example, consider $\operatorname{Spec} R/I$ where $R = \mathbb{C}[x, y]$, $I = (xy)$. We have a picture of $\operatorname{Spec} R$, which is the complex plane, with some mysterious extra "higher-dimensional points". Important algebra fact: The primes of R/I are in bijection with the primes of R containing I. (Here I'm using a prerequisite from Math 210. You should review this fact! This is not a result that you should memorize — you should know why it is true. If you don't remember why it is true, or didn't know this fact, then treat this as an **exercise** and do it yourself.) Thus we can picture $\operatorname{Spec} R/I$ as a subset of $\operatorname{Spec} R$. We have the "0-dimensional points" $(a, 0)$ and $(0, b)$. We also have two "1-dimensional points" (x) and (y).

We get a bit more: the inclusion structure on the primes of R/I corresponds to the inclusion structure on the primes containing I. More precisely, if $J_1 \subset J_2$ in R/I, and K_i is the ideal of R corresopnding to J_i, then $K_1 \subset K_2$.

So the minimal primes of $\mathbb{C}/(xy)$ are the "biggest" points we see, and there are two of them: (x) and (y). Thus we have the intuition that will later be precise: the minimal primes correspond to the "components" of $\operatorname{Spec} R$.

Important example: $\operatorname{Spec} S^{-1}R$ *in terms of* $\operatorname{Spec} R$, where S is a multiplicative subset of R. There are two particularly important flavors of multiplicative subsets. The first is $R - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. (Check that this is a multiplicative set!) The localization $S^{-1}R$ is denoted $R_{\mathfrak{p}}$. Here is a motivating example: $R = \mathbb{C}[x, y]$, $S = R - (x, y)$. The second is $\{1, f, f^2, \dots\}$, where $f \in R$. The localization is denoted R_f. (Notational warning: If (f) is a prime ideal, then $R_f \neq R_{(f)}$.) Here is an example: $R = \mathbb{C}[x, y]$, $f = x$.

Important algebra fact (to review and know): The primes of $S^{-1}R$ are in bijection with the primes of R that *don't meet* the multiplicative set S. So if $S = R - \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal, the primes of $S^{-1}R$ are just the primes of R *contained in* $\mathfrak{p}$. If $S = \{1, f, f^2, \dots\}$, the primes of $S^{-1}R$ are just those primes not containing f (the points where "f doesn't vanish"). A bit more is true: the inclusion structure on the primes of $S^{-1}R$ corresponds to the inclusion structure on the primes not meeting S. (If you didn't know it, take this as an **exercise** and prove it yourself!)

In each of these two cases, a picture is worth a thousand words. In these notes, I'm not making pictures unfortunately. But I'll try to describe them in less than a thousand words.

The case of $S = \{1, f, f^2, \dots\}$ is easier: we just throw out those points where f vanishes. (We will soon call this a *distinguished open set*, once we know what open sets are.) In our example of $R = k[x, y]$, $f = x$, we throw out the y-axis.

Warning: sometimes people are first introduced to localizations in the special case that R is an integral domain. In this case, $R \hookrightarrow R_f$, but this isn't true in general. Here's the definition of localization (which you should be familiar with). The elements of $S^{-1}R$ are of the form r/s where $r \in R$ and $s \in S$, and $(r_1/s_1) \times (r_2/s_2) = (r_1 r_2/s_1 s_2)$, and $(r_1/s_1) + (r_2/s_2) = (r_1 s_2 + s_1 r_2)/(s_1 s_2)$. We say that $r_1/s_1 = r_2/s_2$ if **for some** $s \in S$ $s(r_1 s_2 - r_2 s_1) = 0$.

Example/warning: $R[1/0] = 0$. Everything in $R[1/0]$ is 0. (Geometrically, this is good: the locus of points where 0 doesn't vanish is the empty set, so certainly $D(0) = \operatorname{Spec} R_0$.)

In general, inverting zero-divisors can make things behave weirdly. Example: $R = k[x, y]/(xy)$. $f = x$. What do you get? It's actually a straightforward ring, and we'll use some geometric intuition to figure out what it is. $\operatorname{Spec} k[x, y]/(xy)$ "is" the union of the two axes in the plane. Localizing means throwing out the locus where x vanishes. So we're left with the x-axis, minus the origin, so we expect $\operatorname{Spec} k[x]_x$. So there should be some natural isomorphism $(k[x, y]/(xy))_x \cong k[x]_x$. **Exercise.** Figure out why these two rings are isomorphic. (You'll see that y on the left goes to 0 on the right.)

In the case of $S = R - \mathfrak{p}$, we keep only those primes contained in $\mathfrak{p}$. In our example $R = k[x, y]$, $\mathfrak{p} = (x, y)$, we keep all those points corresponding to "things through the origin", i.e. the 0-dimensional point (x, y), the 2-dimensional point (0), and those 1-dimensional points $(f(x, y))$ where $f(x, y)$ is irreducible and $f(0, 0) = 0$, i.e. those "irreducible curves through the origin". (There is a picture of this in Mumford's Red Book: Example F, Ch. 2, §1, p. 140.)

Caution with notation: If $\mathfrak{p}$ is a prime ideal, then $R_{\mathfrak{p}}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in R$, R_f means you're allowed to divide by f. I find this a bit confusing. Especially when (f) is a prime ideal, and then $R_{(f)} \neq R_f$.

1. Affine schemes II: the underlying topological space

We now define the **Zariski topology** on $\operatorname{Spec} R$. Topologies are often described using open subsets, but it will more convenient for us to define this topology in terms of their complements, closed subsets. If S is a subset of R, define

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} R : S \subset \mathfrak{p}\}.$$

We interpret this as the **vanishing set** of S; it is the set of points on which all elements of S are zero. We declare that these (and no others) are the closed subsets.

1.1. Exercise. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$. This lets us restrict attention to vanishing sets of ideals.

Let's check that this is a topology. Remember the requirements: the empty set and the total space should be open; the union of an arbitrary collection of open sets should be open; and the intersection of two open sets should be open.

1.2. Exercise. (a) Show that $\emptyset$ and $\operatorname{Spec} R$ are both open.
(b) (The intersection of two open sets is open.) Check that $V(I_1 I_2) = V(I_1) \cup V(I_2)$.
(c) (The union of any collection of open sets is open.) If I_i is a collection of ideals (as i runs over some index set), check that $V(\sum_i I_i) = \cap_i V(I_i)$.

1.3. Properties of "vanishing set" function $V(\cdot)$. The function $V(\cdot)$ is obviously inclusion-reversing: If $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$. (Warning: We could have equality in the second inclusion without equality in the first, as the next exercise shows.)

1.4. Exercise. If $I \subset R$ is an ideal, then define its *radical* by

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^{\geq 0}\}.$$

Show that $V(\sqrt{I}) = V(I)$. (We say an *ideal is radical* if it equals its own radical.)

Hence: $V(IJ) = V(I \cap J)$. (Reason: $(I \cap J)^2 \subset IJ \subset I \cap J$.) Combining this with Exercise 1.1, we see

$$V(S) = V((S)) = V(\sqrt{(S)}).$$

1.5. Examples. Let's see how this meshes with our examples from earlier.

Recall that $\mathbb{A}^1_{\mathbb{C}}$, as a set, was just the "old-fashioned" points (corresponding to maximal ideals, in bijection with $a \in \mathbb{C}$), and one "weird" point (0). The Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is not that exciting. The open sets are the empty set, and $\mathbb{A}^1_{\mathbb{C}}$ minus a finite number of maximal ideals. (It "almost" has the cofinite topology. Notice that the open sets are determined by their intersections with the "old-fashioned points". The "weird" point (0) comes along for the ride, which is a good sign that it is harmless. Ignoring the "weird" point, observe that the topology on $\mathbb{A}^1_{\mathbb{C}}$ is a coarser topology than the analytic topology.)

The case $\operatorname{Spec} \mathbb{Z}$ is similar. The topology is "almost" the cofinite topology in the same way. The open sets are the empty set, and $\operatorname{Spec} \mathbb{Z}$ minus a finite number of "ordinary" ((p) where p is prime) primes.

The case $\mathbb{A}^2_{\mathbb{C}}$ is more interesting. I discussed it in a bit of detail in class, using pictures.

1.6. Topological definitions. We'll now define some words to do with the topology.

A topological space is said to be *irreducible* if it is not the union of two proper closed subsets. In other words, X if irreducible if whenever $X = Y \cup Z$ with Y and Z closed, we have $Y = X$ or $Z = X$.

1.7. Exercise. Show that if R is an integral domain, then $\operatorname{Spec} R$ is an irreducible topological space. (Hint: look at the point $[(0)]$.)

A point of a topological space $x \in X$ is said to be *closed* if $\overline{\{x\}} = \{x\}$.

1.8. Exercise. Show that the closed points of $\operatorname{Spec} R$ correspond to the maximal ideals.

Given two points x, y of a topological space X, we say that x is a *specialization* of y, and y is a *generization* of x, if $x \in \overline{\{y\}}$. This now makes precise our hand-waving about "one point contained another". It is of course nonsense for a point to contain another. But it is no longer nonsense to say that the closure of a point contains another.

1.9. Exercise. If $X = \operatorname{Spec} R$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Verify to your satisfaction that this is precisely the intuition of "containment of points" that we were talking about before.

We say that a point $x \in X$ is a *generic point* for a closed subset K if $\overline{\{x\}} = K$.

1.10. Exercise. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

A topological space X is *quasicompact* if given any cover $X = \cup_{i \in I} U_i$ by open sets, there is a finite subset S of the index set I such that $X = \cup_{i \in S} U_i$. Informally: every cover has a finite subcover. This is "half of the definition of quasicompactness". We will like this condition, because we are afraid of infinity.

1.11. Exercise. Show that $\operatorname{Spec} R$ is quasicompact. (Warning: it can have nonquasicompact open sets.)

1.12. Exercise. If X is a finite union of quasicompact spaces, show that X is quasicompact.

Earlier today, we explained that $\operatorname{Spec} R/I$ and $\operatorname{Spec} S^{-1}R$ are naturally subsets of $\operatorname{Spec} R$. All of these have Zariski topologies, and it is natural to ask if the topology behaves well with respect to these inclusions, and indeed it does.

1.13. Exercise. Suppose that $I, S \subset R$ are an ideal and multiplicative subset respectively. Show that $\operatorname{Spec} R/I$ is naturally a closed subset of $\operatorname{Spec} R$. Show that the Zariski topology on $\operatorname{Spec} R/I$ (resp. $\operatorname{Spec} S^{-1}R$) is the subspace topology induced by inclusion in $\operatorname{Spec} R$. (Hint: compare closed subsets.)

1.14. The function $I(\cdot)$, taking subsets of $\operatorname{Spec} R$ to ideals of R. Here is another notion, that is in some sense "opposite" to the vanishing set function $V(\cdot)$. Given a subset $S \subset \operatorname{Spec} R$, $I(S)$ is the ideal of functions vanishing on S. Three quick points: it is clearly an ideal. $I(\overline{S}) = I(S)$. And $I(\cdot)$ is inclusion-reversing: if $S_1 \subset S_2$, then $I(S_2) \subset I(S_1)$.

1.15. Exercise/Example. Let $R = k[x,y]$. If $S = \{(x), (x-1, y)\}$ (draw this!), then $I(S)$ consists of those polynomials vanishing on the y axis, and at the point $(1,0)$. Give generators for this ideal.

More generally:

1.16. Exercise. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S.

1.17. Exercise. Suppose $X \subset \mathbb{A}^3$ is the union of the three axes. Give generators for the ideal $I(X)$.

Note that $I(S)$ is always a radical ideal — if $f \in \sqrt{I(S)}$, then f^n vanishes on S for some $n > 0$, so then f vanishes on S, so $f \in I(S)$.

Here is a handy algebraic fact to know. The *nilradical* $\mathfrak{N} = \mathfrak{N}(R)$ of a ring R is defined as $\sqrt{0}$ — it consists of all functions that have a power that is zero. (Checked that this set is indeed an ideal, for example that it is closed under addition!)

1.18. Theorem. *The nilradical* $\mathfrak{N}(R)$ *is the intersection of all the primes of* R*.*

If you don't know it, then look it up, or even better, prove it yourself. (Hint: one direction is easy. The other will require knowing that any proper ideal of R is contained in a maximal ideal, which requires the axiom of choice.) As a corollary, $\sqrt{I}$ is the intersection of all the prime ideals containing I. (Hint of proof: consider the ring R/I, and use the previous theorem.)

1.19. Exercise. Prove that if $I \subset R$ is an ideal, then $I(V(I)) = \sqrt{I}$.

Hence in combination with Exercise 1.16, we get the following:

1.20. *Theorem. —* $V(\cdot)$ *and* $I(\cdot)$ *give a bijection between closed subsets of* $\operatorname{Spec} R$ *and radical ideals of* R *(where a closed subset gives a radical ideal by* $I(\cdot)$*, and a radical ideal gives a closed subset by* $V(\cdot)$*).*

1.21. Important Exercise. Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} R$ and *prime* ideals of R. From this conclude that in $\operatorname{Spec} R$ there is a bijection between points of $\operatorname{Spec} R$ and irreducible closed subsets of $\operatorname{Spec} R$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset has precisely one generic point.

To drive this point home: Suppose Z is an irreducible closed subset of $\operatorname{Spec} R$. Then there is one and only one $z \in Z$ such that $Z = \overline{\{z\}}$.

2. Distinguished open sets

If $f \in R$, define the **distinguished open set** $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. It is the locus where f doesn't vanish. (I often privately write this as $D(f \neq 0)$ to remind myself of this. I also private call this a D*oesn't vanish set* in analogy with $V(f)$ being the Vanishing set.) We have already seen this set when discussing $\operatorname{Spec} R_f$ as a subset of $\operatorname{Spec} R$.

2.1. Important exercise. Show that the distinguished opens form a base for the Zariski topology.

2.2. Easy important exercise. Suppose $f_i \in R$ for $i \in I$. Show that $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$ if and only if $(f_i) = R$.

2.3. Easy important exercise. Show that $D(f) \cap D(g) = D(fg)$. Hence the distinguished base is a *nice* base.

2.4. Easy important exercise. Show that if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n.

2.5. Easy important exercise. Show that $f \in \mathfrak{N}$ if and only if $D(f) = \emptyset$.

We have already observed that the Zariski topology on the distinguished open $D(f) \subset \operatorname{Spec} R$ coincides with the Zariski topology on $\operatorname{Spec} R_f$.

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