18.034 MIDTERM 2

Explain your answers clearly; show all steps. Calculators may not be used. All problems have equal value. Please put your name on every sheet. Good luck!

1. (a) $y_1$, $y_2$, and $y_3$ are 3 solutions of the differential equation $(1-t)y'''+y''+t^2y'+t^3y=0$ on the interval $1<t<\infty$. Calculate the function $W(y_1, y_2, y_3)(t)$ given that $W(y_1, y_2, y_3)(2) = 3$.

(b) The equation $y'+a(x)y=0$ has for a solution
$$\phi(x) = e^{-\int_{x_0}^x a(t)dt}.$$ (Here let $a$ be continuous on an interval $I$ containing $x_0$.) This suggests trying to find a solution of
$$L(y) = y'' + a_1(x)y' + a_2(x)y = 0$$
of the form
$$\phi(x) = e^{\int_{x_0}^x p(t)dt}$$
where $p$ is a function to be determined. Show that $\phi$ is a solution of $L(y) = 0$ if and only if $p$ satisfies the first-order non-linear equation $y' = -y^2 - a_1(x)y - a_2(x)$. (Remark: This last equation is called a Riccati equation.)

2. (a) Consider the equation $y'' - \frac{2}{x^2}y = 0$ (for $0 < x < \infty$). Find all solutions. (Hint: Try functions of the form $y = x^r$. How do you know you’ve found all the solutions?)

(b) Find all solutions to the equation $y'' - \frac{2}{x^2}y = x$. Hint: Use “variation of parameters”. Suppose $\phi_1$ and $\phi_2$ are linearly independent solutions to the homogeneous version of the equation (see (a)). Look for a solution of the form $\phi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$.

3. Iterate $x \to \sqrt{1+x}$. Start with $x = 0$. What happens?

4. (a) State the Existence and Uniqueness Theorem for differential equations of the form $y' = f(x,y)$.

(b) Consider the differential equation $y' = t^2(y+1)$ on the interval $\mathbb{R}$, with initial condition $y(0) = 0$. Find a solution $y = \phi(t)$ defined for all $t \in \mathbb{R}$. If the first few Picard iterates (used in the proof of the Existence and Uniqueness Theorem described in (a)) are $\phi_0(t) = 0$, $\phi_1(t)$, $\phi_2(t)$, find $\phi_1(t)$ and $\phi_2(t)$.

(c) Explain why the $\phi_1(t)$ and $\phi_2(t)$ you found are approximations to $\phi(t)$.

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5. Consider the equation \( y'' + \cos(x)y' + \sin(x)y = 0 \).

(a) Let \( \phi(x) \) be a nontrivial solution, and let \( \psi(x) = \phi(x + 2\pi) \). Prove that \( \psi(x) \) is also a solution.

(b) Show that \( \phi(x) \) is a periodic solution of period \( 2\pi \) if, and only if, \( \phi(0) = \phi(2\pi) \) and \( \phi'(0) = \phi'(2\pi) \).

(c) Let \( \phi_1(x) \), \( \phi_2(x) \) be two solutions satisfying \( \phi_1(0) = 1 \), \( \phi_1'(0) = 0 \), \( \phi_2(0) = 0 \), \( \phi_2'(0) = 1 \). Show that there are constants \( a \) and \( b \) such that

\[
\phi_1(x + 2\pi) = a\phi_1(x) + b\phi_2(x).
\]

(Hint: See (a).)

6. Let \( A = \begin{pmatrix} 3 & 1 \\ -5 & -3 \end{pmatrix} \). Find the eigenvalues of \( A \). Find eigenvectors of \( A \) corresponding to each of the eigenvalues. Calculate \( A^{2000} \).