18.024 QUIZ IV SOLUTIONS

1. Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field defined in all of $\mathbb{R}^2$ except the origin. Assume $Q_x - P_y = \frac{1}{x^2+y^2}$. You are given that $\oint_{C_1} \vec{F} \cdot d\vec{\alpha} = 5$, where C_1 is the unit circle centered at the origin, directed counterclockwise. Find $\oint_{C_2} \vec{F} \cdot d\vec{\alpha}$, if C_2 is the circle of radius 2 centered at the origin, oriented counterclockwise.

Solution. By Green's theorem,

$$\begin{aligned} \oint_{C_2} \vec{F} \cdot d\vec{\alpha} &= \oint_{C_1} \vec{F} \cdot d\vec{\alpha} + \iint_R \frac{1}{r^2}\, dxdy \\ &= 5 + \int_0^{2\pi} \int_1^2 1/r\, drd\theta \\ &= \mathbf{5 + 2}\pi \ln \mathbf{2} \end{aligned}$$

2. Find the area of the region bounded by the curve $4x^2 - 4xy + y^2 - 2y = 0$ and $y = 1$. (Hint: use a suitable linear transformation.)

Solution. Let $u = 2x - y$, $v = y$, so $x = (u + v)/2$, $y = v$, and $\partial(x, y)/\partial(u, v) = 1/2$. The region corresponds to the area $-\sqrt{2} \le u \le \sqrt{2}$, $u^2/2 \le v \le 1$, which has area

$$\begin{aligned} \int_{-\sqrt{2}}^{\sqrt{2}} \left(1 - u^2/2\right)\, du &= (u - u^3/6)^{\sqrt{2}})_{-\sqrt{2}} \\ &= 2\left(\sqrt{2} - 2\sqrt{2}/6\right) \\ &= \frac{\sqrt{2}}{3}(3 \cdot 2 - 2) \\ &= 4\sqrt{2}/3. \end{aligned}$$

Answer: $4\sqrt{2}/3 \times 1/2 = \mathbf{2}\sqrt{\mathbf{2}}/\mathbf{3}$.

3. Find the area of the portion of the surface $z = xy + 4$ lying in the cylinder $x^2 + y^2 = a^2$.

Solution.

$$\begin{aligned} \iint_{x^2+y^2 \le a^2} \sqrt{1 + x^2 + y^2}\, dxdy &= \int_0^{2\pi} \int_0^a \sqrt{1 + r^2} r\, drd\theta \\ &= 2\pi \left(\frac{1}{3}(r^2 + 1)^{3/2}\right)_0^a \\ &= \mathbf{2}\pi/\mathbf{3}\left((\mathbf{a^2 + 1})^{\mathbf{3/2}} - \mathbf{1}\right). \end{aligned}$$

Date: Spring 2001.

4. Calculate the hypervolume of a four-dimensional sphere $w^2+x^2+y^2+z^2 \le R^2$. You may use the following integrals:

$$\int_0^{\pi/2} \sin^2\theta\, d\theta = \int_0^{\pi/2} \cos^2\theta\, d\theta = \pi/4,$$
$$\int_0^{\pi/2} \sin^4\theta\, d\theta = \int_0^{\pi/2} \cos^4\theta\, d\theta = 3\pi/16,$$
$$\int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta = \pi/16.$$

Solution.

$$\begin{aligned}
\int_0^{2\pi}\int_0^R\int_0^\pi 2\sqrt{R^2-\rho^2}\rho^2 \sin\phi\, d\phi d\rho d\theta &= 2\pi \times 2 \times 2 \times \int_0^R \sqrt{R^2-\rho^2}\rho^2\, d\rho \\
&= 8\pi \int_0^{\pi/2} R\cos\alpha \times R^2 \sin^2\alpha \times R\cos\alpha\, d\alpha \\
&= 8\pi R^4 \int_0^{\pi/2} \sin^2\alpha \cos^2\alpha\, d\alpha \\
&= \boldsymbol{\pi^2 R^4/2}.
\end{aligned}$$

5. Let C be the intersection of the plane $ax+by+3z=0$ with the cylinder $x^2+y^2=c$, oriented counterclockwise as viewed from a point high on the z-axis. Use Stokes' theorem to evaluate the integral $\oint_C \vec{F}\cdot d\vec{\alpha}$ where $\vec{F}$ is the vector field

$$\vec{F} = (y-\sin x)\vec{i} + (y^{10}-3z)\vec{j} + (4x-2y+17z)\vec{k}.$$

Solution. $\vec{\nabla}\times\vec{F} = \vec{i} - 4\vec{j} - \vec{k}$. Do the integral over the flat (tilted) oval, using Stokes.

$$\vec{\nabla}\times\vec{F}\cdot\vec{n} = (1,-4,-1)\cdot(a,b,3)/\sqrt{a^2+b^2+9} = (a-4b-3)/\sqrt{a^2+b^2+9}.$$

The answer is

$$\pi c \cdot \sqrt{a^2+b^2+9}/3 \cdot (a-4b-3)/\sqrt{a^2+b^2+9} = \boldsymbol{\pi c(a-4b-3)/3}.$$

6. Suppose $\vec{F} = y\vec{i} - x\vec{j} + z\vec{k}$. Let S be that portion of the surface $z=f(x,y)$ contained in the cylinder $x^2+y^2\le 9$, and let $\vec{n}$ be the upward normal. Find the relationship between the total flux upward through S $\iint_S \vec{F}\cdot\vec{n}\, dA$ and the average value of f on the disc $x^2+y^2\le 9$. (Hint: Use the divergence theorem on the three-dimensional region bounded by S, the cylinder $x^2+y^2=9$, and the disc $z=0$, $x^2+y^2\le 9$.) You may assume that $f(x,y)$ is continuously differentiable and non-negative.

Solution.

$$\iiint \vec{\nabla}\cdot\vec{F}\, dV = \iint_{x^2+y^2\le 9} f(x,y) = \textbf{average of } f \times \boldsymbol{9}\pi.$$