SKETCHES OF SOLUTIONS TO 18.024
PRACTICE QUIZ II

1. (16 points) Consider the curve given in polar coordinates by $r = e^{-t}$, $\theta = t$ for $0 \leq t \leq 2M\pi$ ($M$ a positive integer).

(a) Sketch this curve when $M = 2$.
(b) Find the length of this curve for general $M$. What happens as $M$ becomes large?

Solution. (a) Your sketch should wind inwards counterclockwise around the origin twice. (b) Use the formula for $\vec{v}$ (given at the top of the practice quiz) to get

$$\int_0^{2M\pi} \sqrt{(r')^2 + (r'')^2} \, dt = \int_0^{2M\pi} \sqrt{e^{-2t} + e^{-2t}} \, dt = \sqrt{2} \left( 1 - e^{-2M\pi} \right).$$

As $M \to \infty$, the length goes to $\sqrt{2}$.

2. (16 points) A particle moves along a curve $C$ in space. Its acceleration vector has constant length 3 and its speed at time $t \geq 0$ is $1/(1 + 2t)$. Find the curvature of the curve in terms of $t$.

Solution. Use the formula

$$\vec{a} = v' \vec{T} + \kappa v^2 \vec{N}$$

(the second equation given at the top of the practice quiz) from which

$$||\vec{a}||^2 = (v')^2 + \kappa^2 v^4.$$

This gives

$$\kappa = \sqrt{36t^2 + 36t + 5}.$$

3. (20 points)

(a) Complete the definition. A vector-valued function

$$\vec{f} : S \to \mathbb{R}^3$$

where $S \subset \mathbb{R}^2$ contains a ball $B(\vec{a}; r)$ of radius $r$ around $\vec{a} \in \mathbb{R}^2$ is said to be differentiable at $\vec{a}$ if for all $\vec{v} \in \text{[blank]}$,

$$\vec{f}(\vec{a} + \vec{v}) = \vec{f}(\vec{a}) + T_{\vec{a}} \vec{v} + ||\vec{v}|| E_{\vec{a}}(\vec{v})$$

where [blank]. (Hint: the second blank is a fact about the function $E_{\vec{a}}$.)

(b) Let $f(x, y)$ be a function defined in $\mathbb{R}^2$ (the plane). Answer the following questions “yes” or “no” (+3 points for each correct answer, -3 for each incorrect answer).

Date: Spring 2001.
(i) Suppose that $D_1f$ and $D_2f$ exist at $(0,0)$. Does it follow that $f$ is continuous at $(0,0)$? Y/N Does it follow that the functions $g(t) = f(t,0)$ and $h(t) = f(0,t)$ are continuous at $t = 0$? Y/N

(ii) Suppose that $D_1f$ and $D_2f$ exist in a neighborhood of $(0,0)$ and are continuous at $(0,0)$. Does it follow that $f$ is continuous at $(0,0)$? Y/N Does it follow that $f'(0;\vec{y})$ exists for all $\vec{y}$? Y/N

**Solution.**

(a) $\vec{v} \in B(\vec{0}; r)$ (not $B(\vec{a}; r)$ — do you see why?): $E_\vec{a}(\vec{v}) \to 0$ as $\vec{v} \to \vec{0}$ (remember to write $0$ not $0$ for the zero vector). (b) (i) NY (b) YY.

4. (16 points)

(a) Suppose $f(x,y,z)$ is a differentiable scalar-valued function such that $f(1,1,1) = 2$, and the $\nabla f(1,1,1) = (3,4,5)$. Find the (Cartesian) equation of the tangent plane to the level surface $f(x,y,z) = 2$ at $(x,y,z) = (1,1,1)$.

(b) Suppose $f(u,v)$ is a differentiable scalar-valued function such that $f(1,1) = 2$, and $\nabla f(1,1) = (3,4)$. Find the (Cartesian) equation of the tangent plane to the graph of $f$ (i.e. $z = f(x,y)$) when $(x,y) = (1,1)$.

(c) Show that $f(x,y) = (\sin x)(\sin y)$ has a critical point (i.e. the gradient is $\vec{0}$) at $(x,y) = (0,0)$. Does $f$ have a minimum, maximum, or saddle point here?

**Solution.**

For (a) and (b), use the gradient. (a) $3(x-1)+4(y-1)+5(z-1) = 0$, or $3x+4y+5z = 12$ (either is acceptable of course). (b) $3(x-1)+4(y-1)-(z-2) = 0$ or $(z-2) = 3(x-1)+4(y-1)$ or $z = 3x+4y-5$. (c) Saddle points (using second derivative test). (You can also see this geometrically: which direction can you walk from $(0,0)$ so $f$ decreases? Increases?)

5. (16 points) Given a differentiable function $F(u,v)$, consider the composite function $f(x,y) = F(3x-y,2x-y)$. Find $\frac{\partial f}{\partial x}(1,1)$ if $D_1F(1,1) = -4, D_1F(2,1) = 7, D_2F(1,1) = 3, D_2F(2,1) = -3$.

**Solution.**

Use the chain rule. Let $u = 3x - y$, $v = 2x - y$. Then

$$\frac{\partial f}{\partial x}(1,1) = \frac{\partial F}{\partial u}(u = 2, v = 1) \frac{\partial u}{\partial x}(x = 1, y = 1) + \frac{\partial F}{\partial v}(u = 2, v = 1) \frac{\partial v}{\partial x}(x = 1, y = 1)$$

$$= 7 \times 3 + (-3) \times 2$$

$$= 15.$$

(Notice the red herring in the problem: we never use $D_1F(1,1)$!)

6. (16 points) The equation $x^2 + z^3 + yz = 3$ defines $z$ implicitly as a function of $x$ and $y$, say $z = f(x,y)$. Find $\frac{\partial f}{\partial x}$ in terms of $x$, $y$, and $z$. Find $\frac{\partial^2 f}{\partial x^2}$ in terms of $x$, $y$, $z$, and $\frac{\partial f}{\partial x}$.

**Solution.**

Use implicit differentiation. Differentiate $x^2 + f(x,y)^3 + yf(x,y) = 3$ with respect to $x$ to get

$$2x + 3f^2f_x + yf_x = 0$$
from which
\[ f_x = \frac{-2x}{3f^2 + y} = \frac{-2x}{3z^2 + y}. \]

Differentiate this again with respect to \( x \) to get
\[ f_{xx} = \frac{-2(3f^2 + y) - (-2x)(6ff_x)}{(3f^2 + y)^2} = \frac{-2(3z^2 + y) - (-2x)(6zf_x)}{(3z^2 + y)^2}. \]

If you wanted, you could substitute the formula for \( f_x \) (in terms of \( x, y, \) and \( z \)) into this one, but it would be ugly, and nothing much would be gained.)