Here are some formulas (for reference) for velocity $\vec{v}$ and acceleration $\vec{a}$.

In terms of $\vec{T}$ and $\vec{N}$:

$$\vec{v} = \left( \frac{ds}{dt} \right) \vec{T},$$

$$\vec{a} = \left( \frac{d^2 s}{dt^2} \right) \vec{T} + \kappa \left( \frac{ds}{dt} \right)^2 \vec{N}.$$ 

In polar coordinates, in terms of $\vec{u}_r$ and $\vec{u}_\theta$:

$$\vec{v} = \left( \frac{dr}{dt} \right) \vec{u}_r + \left( r \frac{d\theta}{dt} \right) \vec{u}_\theta,$$

$$\vec{a} = \left( \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right) \vec{u}_r + \left( r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right) \vec{u}_\theta.$$ 

1. (16 points) Consider the curve given in polar coordinates by $r = e^{-t}$, $\theta = t$ for $0 \leq t \leq 2M\pi$ ($M$ a positive integer).

(a) Sketch this curve when $M = 2$.

(b) Find the length of this curve for general $M$. What happens as $M$ becomes large?

2. (16 points) A particle moves along a curve $C$ in space. Its acceleration vector has constant length 3 and its speed at time $t$ is $1/(1 + 2t)$. Find the curvature of the curve in terms of $t$.

3. (20 points)

(a) Complete the definition. A vector-valued function

$$\vec{f} : S \to \mathbb{R}^3$$

where $S \subset \mathbb{R}^2$ contains a ball $B(\vec{a}; r)$ of radius $r$ around $\vec{a} \in \mathbb{R}^2$ is said to be differentiable at $\vec{a}$ if for all $\vec{v} \in [\text{blank}]$,

$$\vec{f}(\vec{a} + \vec{v}) = \vec{f}(\vec{a}) + T_a \vec{v} + |\vec{v}| E_{\vec{a}}(\vec{v})$$

where [blank]. (Hint: the second blank is a fact about the function $E_{\vec{a}}$.)

(b) Let $f(x, y)$ be a function defined in $\mathbb{R}^2$ (the plane). Answer the following questions “yes” or “no” (+3 points for each correct answer, −3 for each incorrect answer).
(i) Suppose that $D_1 f$ and $D_2 f$ exist at $(0,0)$. Does it follow that $f$ is continuous at $(0,0)$? Y/N Does it follow that the functions $g(t) = f(t,0)$ and $h(t) = f(0,t)$ are continuous at $t = 0$? Y/N

(ii) Suppose that $D_1 f$ and $D_2 f$ exist in a neighborhood of $(0,0)$ and are continuous at $(0,0)$. Does it follow that $f$ is continuous at $(0,0)$? Y/N

Does it follow that $f_0(\tilde{0};\tilde{y})$ exists for all $\tilde{y}$? Y/N

4. (16 points)

(a) Suppose $f(x,y,z)$ is a differentiable scalar-valued function such that $f(1,1,1) = 2$, and the $\nabla f(1,1,1) = (3,4,5)$. Find the (Cartesian) equation of the tangent plane to the level surface $f(x,y,z) = 2$ at $(x,y,z) = (1,1,1)$.

(b) Suppose $f(x,y)$ is a differentiable scalar-valued function such that $f(1,1) = 2$, and $\nabla f(1,1) = (3,4)$. Find the (Cartesian) equation of the tangent plane to the graph of $f$ (i.e. $z = f(x,y)$) when $(x,y) = (1,1)$.

(c) Show that $f(x,y) = (\sin x)(\sin y)$ has a critical point (i.e. the gradient is $\tilde{0}$) at $(x,y) = (0,0)$. Does $f$ have a minimum, maximum, or saddle point here?

5. (16 points) Given a differentiable function $F(u,v)$, consider the composite function $f(x,y) = F(3x-y,2x-y)$. Find $\frac{\partial f}{\partial x}(1,1)$ if $D_1 F(1,1) = -4$, $D_1 F(2,1) = 7$, $D_2 F(1,1) = 3$, $D_2 F(2,1) = -3$.

6. (16 points) The equation $x^2 + z^3 + yz = 3$ defines $z$ implicitly as a function of $x$ and $y$, say $z = f(x,y)$. Find $\frac{\partial f}{\partial x}$ in terms of $x, y,$ and $z$. Find $\frac{\partial^2 f}{\partial x^2}$ in terms of $x, y, z,$ and $\frac{\partial f}{\partial x}$.

Office hours this week: Wednesday 3-5.