COMPLEX ALGEBRAIC SURFACES CLASS 9

RAVI VAKIL

CONTENTS

(At the end of last lecture I discussed the Weak Factorization Theorem, Resolution of Singularities, and de Jong's Alternation Theorem.)

Recap of last time. Proof of the universal property of blowing up.

Exercise. The blow-up $\mathrm{Bl}_p S \to S$ has a universal property in the "other direction": every morphism f from $\mathrm{Bl}_p S$ to a variety X to a variety X that contracts E to a point factors through S. (Hint, basically proof: reduce first to the case X affine, then $X = \mathbb{A}^n$, then to $X = \mathbb{A}^1$, so we're talking about functions on $S - \{p\}$. But every function on $S - \{p\}$ extends over S.)

Minimal surfaces: those with no (-1)-curves.

To understand surfaces up to birational or biregular invariance, we should focus on minimal surfaces.

Theorem (Castelnuovo's contractibility criterion). Let S be a surface and $E \subset S$ a rational curve with $E^2 = -1$.

I won't prove this in its entirety, but will say the strategy. We'll produce a line bundle that will give a map to projective space that will preserve $S - E$, but collapse E to a point. Then the harder part is to show that the point is a smooth point of the new surface. I'll omit that: the proofs involve some infinitesimal analysis beyond what I claimed are the prerequisites of the course.

1. CONSTRUCTION OF CASTELNUOVO'S CONTRACTION MAP

Choose a very ample divisor on S. By taking an appropriate multiple, we may assume that $H^1(S, \mathcal{O}(H)) = 0$ by Serre vanishing. Suppose $k = H \cdot E$, and consider the line bundle $\mathcal{O}(H + kE)$. Note that $(H + kE) \cdot E = 0$; this is the motivation for taking this divisor.

Date: Wednesday, October 30.

We'll now show that $\mathcal{O}(H+kE)$ is basepoint free (and hence gives a map to projective space), and that it just collapses E and keeps the rest of the surface the way it was.

First, we'll show that $H^1(X,\mathcal{O}(H+kE))=0$. Fix a section t of $\mathcal{O}_S(E)$ vanishing along E. Take the exact sequence

$$0 \to \mathcal{O}_S(-E) \xrightarrow{t} \mathcal{O}_S \to \mathcal{O}_E \to 0.$$

The morphism labeled t means "multiply by t". Twist this by $H+iE$ to get

$$0 \to \mathcal{O}_S(H+(i-1)E) \xrightarrow{t} \mathcal{O}_S(H+iE) \to \mathcal{O}_E(k-i) \to 0.$$

By Serre duality, $H^1(E,\mathcal{O}_E(r)) = H^0(E,\mathcal{O}_E(-2-r))$, which is 0 for $r \geq 0$, so taking the long exact sequence we get:

$$0 \to H^0(S,\mathcal{O}_S(H+(i-1)E)) \xrightarrow{t} H^0(S,\mathcal{O}_S(H+iE)) \to H^0(E,\mathcal{O}_E(k-i))$$

$$\to H^1(S,\mathcal{O}_S(H+(i-1)E)) \xrightarrow{t} H^1(S,\mathcal{O}_S(H+iE)) \to 0.$$

for $1 \leq i \leq k$. By induction on i, $H^1(S,\mathcal{O}_S(H+iE))=0$ for $1 \leq i \leq k$, so we have:

$$0 \to H^0(S,\mathcal{O}_S(H+(i-1)E)) \xrightarrow{t} H^0(S,\mathcal{O}_S(H+iE)) \to H^0(E,\mathcal{O}_E(k-i)) \to 0.$$

So we can build up $H^0(S,\mathcal{O}_S(H+kE))$ inductively. Choose a basis $s_0, \dots, s_n$ of $H^0(S,\mathcal{O}_S(H))$. Then using $i=1$, the sections of $H^0(S,\mathcal{O}_S(H+E))$ correspond to

$$ts_0,\dots,ts_n, \text{ plus } k \text{ terms coming from } H^0(E,\mathcal{O}_E(k-1)).$$

Using $i=2$, the sections of $H^0(S,\mathcal{O}_S(H+2E))$ correspond to

$$t^2s_0,\dots,t^2s_n, \text{ plus } k \text{ terms } \times t, \text{ plus } k-1 \text{ terms coming from } H^0(E,\mathcal{O}_E(k-2)).$$

Keep doing this k times to get:

$$t^ks_0,\dots,t^ks_n, \text{ plus } k \text{ terms } (\times t^{k-1}) \text{ plus } k-1 \text{ terms } (\times t^{k-2}) \text{ plus} \cdots \text{ plus } 1 \text{ term } (\times t^0)$$

By looking just at the first $n+1$ terms, we see that there are no basepoints away from $(t=0)=E$. For a point on E, all the terms are non-zero except for the last one. Thus we have a basepoint free map, that sends E to $[0;\cdots;0;1]$. By looking at the first $n+1$ terms, we see that it is an embedding everywhere else: it separates points and tangent vectors.

That completes the construction of the map. However, we can see the blow-up as the resolution of the projection from $[0;\cdots;0;1]$...

The proof uses infinitesimal analysis, and I'll omit it. (If time, give them a sketch. The image surface is smooth at the key point $p=[0;\dots;0;1]$, if $\mathfrak{m}/\mathfrak{m}^2$ has dimension 2 as a k-vector space, i.e. $\mathcal{O}/\mathfrak{m}^2$ has dimension 3. It is singular if the dimension is bigger than 3. We can pull back this scheme in the target, by intersecting $x_0^2=0, \dots, x_{n-1}^2=0$. We can pull back these equations to our original surface, to get as an equation the square of the ideal E. This has a 3-dimensional space of sections. All that needs to be checked is that the pullback of the first-order neighborhood of p really *is* the dimension of the pre-image scheme; this requires infinitesimal analysis.)

2. RULED SURFACES

A surface is *ruled* if it is birationally equivalent to $C \times \mathbb{P}^1$, where C is a smooth curve. (More generally, a variety is ruled if through every curve, there is a $\mathbb{P}^1$.)

Examples:

1. $C \times \mathbb{P}^1$.
2. $\mathbb{P}^2$.
3. a $\mathbb{P}^1$-bundle over C (i.e. something that Zariski-locally on C is isomorphic to $\mathbb{P}^1 \times C$).
4. Special case of above: if E is a rank 2 vector bundle over C, then its projectivization is a ruled surface over C. Informal description. In fact, we'll see that this is the same as (3) above. Precise definition: $\mathcal{F}$ is a rank 2 locally free sheaf. Then the corresponding ruled surface is
$$\operatorname{Proj}_C \oplus_{i=0}^{\infty} \operatorname{Sym}^i \mathcal{F}.$$

Question over $\mathbb{C}$: Suppose you had a surface $S \to C$ such that the fibers are all isomorphic to $\mathbb{CP}^1$. (This is called *geometrically ruled*.) Is S necessarily a $\mathbb{P}^1$-bundle over C?

Those of type (3) above are geometrically ruled.

An important example are the following rational (geometrically) ruled surfaces over $\mathbb{P}^1$, sometimes called *Hirzebruch surfaces*: the projectivization of the locally free sheaves $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$, where $n \geq 0$. Called $\mathbb{F}_n$. In fancy notation $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. We'll see in a class or so that these are all different, and these are all the geometrically ruled surfaces over $\mathbb{P}^1$.

You may naively think that geometrically ruled is the same as (3) ($\mathbb{P}^1$-bundles). And you would be right, but it is not obvious. Here is an example showing that your intuition may be wrong.

Example. There is a map from a threefold to a surface that is geometrically ruled but not of form (3). Instead, I'll give an example a sixfold fibered over a fivefold.

Consider the $\mathbb{P}^5$ parametrizing plane conics, with point $[a_{x^2}; a_{xy}; a_{y^2}; a_{xz}; a_{yz}; a_{z^2}]$ corresponding to conic $a_{x^2}x^2 + \cdots + a_{z^2}z^2 = 0$ in $\mathbb{P}^2$. There is a universal conic:

$$\begin{array}{ll} U & = \{([x;y;z],[a_{x^2};\cdots;a_{z^2}]) : a_{x^2}x^2 + \cdots + a_{z^2}z^2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^5 \\ \downarrow & \\ \mathbb{P}^5 & \end{array}$$

Throw out the locus in $\mathbb{P}^5$ corresponding to reducible conics:

$$\begin{array}{ccccc} U' & \to & U & \subset \mathbb{P}^2 \times \mathbb{P}^5 \\ \downarrow & & \downarrow & \\ \mathbb{P}^5 - \Delta & & \mathbb{P}^5 & \end{array}$$

Fact: This isn't a trivial $\mathbb{P}^1$-bundle over any Zariski-open set. Argument is short, and if you're curious, ask me about it. Key idea: if it were, then there would be a divisor intersecting the class of a fiber with multiplicity 1. (Describe why.) We can work out the topology of U explicitly, and show that any divisor meets the class of a fiber with even multiplicity.

More strongly: if you restrict to the generic point, i.e. to the function field of $\mathbb{P}^5$, then you don't get $\mathbb{P}^1$ over that field.

Arithmetic version of that same comment. Those of you who are more arithmetically inclined will be less surprised by this example. (Skip in class.) For example, any conic in the complex plane is isomorphic to the projective line, but over a trickier field such as $\mathbb{Q}$ this isn't true. For example, $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_R$ is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$. This is just an artifact of working over a field that isn't algebraically closed. And this example shows that even if you think you only care about algebraically closed fields, such as $\mathbb{C}$, then you may still come across non-algebraically closed fields, such as function fields of curves, or varieties in general.

Noether-Enriques Theorem. Suppose $\pi : S \to C$ is geometrically ruled. Then S is of type (3) above, i.e. it is the projectivization of some rank 2 invertible sheaf / vector bundle.

Slightly more generally: Suppose $\pi : S \to C$, and $x \in C$ such π is smooth over C and $\pi^{-1}(x)$ is isomorphic to $\mathbb{P}^1$. Then there is a Zariski-open subset $U \subset C$ containing x and a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \stackrel{\sim}{\to} & U \times \mathbb{P}^1 \\ & \stackrel{\pi}{\searrow} \quad \swarrow & \\ & U. & \end{array}$$

Proof. Three-step proof. The key step is 2, where we produce a divisor H of S meeting the class of a fiber with multiplicity 1.

Step 1: $H^2(S, \mathcal{O}_S) = 0$. Let F be the class of $\pi^{-1}(x)$. $F^2 = 0$. By the genus formula, $-2 = F \cdot K$. Suppose otherwise that $H^2(S, \mathcal{O}_S) > 0$, so by Serre duality $\mathcal{K}$ has a section. Let K be the divisor associated to the zero set of that section. Then $K = nF + K'$, where K' has no F-component. Finally, $-2 = K \cdot F = (K' + nF) \cdot F \geq 0$, contradiction.

Step 2: There is a divisor H of S such that $H \cdot F = 1$. Here we work over $\mathbb{C}$. There is another proof in Hartshorne Chapter V which works over any field. Recall that from the long exact sequence for $0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 1$, we have $\operatorname{Pic}(S) \to H^2(S, \mathbb{Z}) \to H^2(S, \mathcal{O}_S) = 0$ using Step 1. Thus it suffices to find a class $h \in H^2(S, \mathbb{Z})$ with $h \cdot f = 1$, where f is the image of F in $H^2(S, \mathbb{Z})$.

Consider the map of $\mathbb{Z}$-modules $H^2(S, \mathbb{Z}) \to \mathbb{Z}$ given by $a \mapsto a \cdot f$. The image is an ideal of $\mathbb{Z}$, say $d\mathbb{Z}$ for $d \geq 0$. Clearly non-zero: take a very ample. We want to show that $d = 1$.

The map $a \mapsto (a \cdot f)/d$ is a linear form on $H^2(S, \mathbb{Z})$. By Poincare duality, $H^2(S, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \to H^4(S, \mathbb{Z}) = \mathbb{Z}$ is a duality, i.e. $H^2(S, \mathbb{Z}) \to \operatorname{Hom}(H^2(S, \mathbb{Z}), \mathbb{Z})$ is a surjective (with kernel equal to the torsion subgroup). Thus there is an element $f' \in H^2(S, \mathbb{Z})$ such that $a \cdot f = (a \cdot f)/d$ for all $a \in H^2(S, \mathbb{Z})$. If $d > 1$, this class is a bit hard to imagine; let's get a contradiction.

Let k be the image of K in $H^2(S, \mathbb{Z})$. I claim that for any $b \in H^2(S, \mathbb{Z})$, $F(b) = b^2 + b \cdot k$ is always even. Proof: it is always an integer. It's even for all irreducible curves. And it is additive modulo 2, i.e. $F(b_1 + b_2) \equiv F(b_1) + F(b_2) \pmod 2$.

Thus the following number is even: $f' \cdot f' + f' \cdot k = f \cdot f/4 + f \cdot k/2 = -2/d$, and we're done.

Interesting consequence we'll us later: there are no multiple fibers. (Draw picture.) (A multiple fiber is a multiple of an irreducible F' by an integer greater than 1.) Reason: there is a divisor class meeting it with multiplicity 1

Step 3: next day.