COMPLEX ALGEBRAIC SURFACES CLASS 8

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CONTENTS

Recap of last time. Strict and proper transforms.

If $\pi : S' \to S$ is a blow up of S at p, with exceptional curve $E \subset S'$, D and D' be divisors on S. Then $\pi^* D \cdot \pi^* D' = D \cdot D'$, $E \cdot \pi^* D = 0$, $E^2 = -1$. There is an isomorphism $\operatorname{Pic} S \oplus \mathbb{Z} \xrightarrow{\sim} \operatorname{Pic} S'$ defined by $(D, n) \mapsto \pi^* D + nE$. The same with Pic replaced by NS. $K_{S'} = \pi^* K_S + E$.

Rational maps of surfaces, linear systems, and elimination of indeterminacy. Base points. Fixed part of a linear system: union of fixed (dimension 1) components. If the linear systems has no fixed part, then it has only a finite number of fixed points.

There is a bijection between:

(i) $\{$ rational maps $\pi : S \dashrightarrow \mathbb{P}^n$ such that $\pi(S)$ is contained in no hyperplane $\}$

(ii) $\{$ linear systems on S without fixed part and of dimension n $\}$

Theorem (Elimination of indeterminacy). Let $\pi : S \dashrightarrow X$ be a rational map from a surface to a projective variety. Then there exists a surface S', a morphism $\eta : S' \to S$ which is the composite of a finite number of blow-ups, and a morphism $f : S' \to X$ such that the diagram

$$\begin{tikzcd} & S' \arrow[dl, "\eta"'] \arrow[dr, "f"] & \\ S \arrow[rr, dashed, "\pi"] & & X \end{tikzcd}$$

is commutative.

Idea: blow up fixed points, show that D^2 decreases.

Theorem (Universal property of blowing up). Let $f : X \to S$ be a birational morphism of surfaces, and suppose that the rational map f^{-1} is undefined at a point p of S. Then f

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factors as

$$f : X \xrightarrow{g} \mathrm{Bl}_p S \xrightarrow{\pi} S$$

where g is a birational morphism and π is the blow-up at p.

Proof: Today.

Applications of the universal property of blowing up:

Theorem (all birational morphisms factor into blow-ups). Let $f : S \to S_0$ be a *birational* morphism of surfaces. Then there is a sequence of blow-ups $\pi_k : S_k \to S_{k-1}$ ($k - 1, \dots, n$) and an isomorphism $u : S \xrightarrow{\sim} S_n$ such that $f = \pi_1 \circ \cdots \circ \pi_n \circ u$.

Theorem (all birational maps can be factored into blow-ups). Let $\phi : S \dashrightarrow S'$ be a birational map of surfaces. Then there is a surface S'' and a commutative diagram

$$\begin{tikzcd} & S'' \arrow[dl, "f"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "\phi"] & & S' \end{tikzcd}$$

where the morphisms f and g are composites of blow-ups.

1. Proof of the universal property of blowing up

This will be pretty tricky.

Lemma 1. Let S be a surface, possibly singular, S' a surface, and $f : S \to S'$ a birational morphism. Suppose that the rational map f^{-1} is undefined at a point $p \in S'$. Then $f^{-1}(p)$ is a curve on S (and there are no isolated points). (Two uses of f^{-1}!)

Proof. We may reduce to an affine neighborhood U of S, meeting $f^{-1}(p)$. Then there is an embedding of this neighborhood $U \hookrightarrow A^n$ (with coordinates $(x_1, \dots)$ on $\mathbb{A}^n$). So we have a rational map $S' \dashrightarrow \mathbb{A}^n$ given by rational functions $g_1, \dots, g_n$. Now f^{-1} is undefined at p, so one of these functions is undefined, say g_1. Then $g_1 = u/v$ where u and v are defined locally on S', where u and v have no common factor. (Here we reduce to an open set on S'. Make it smaller so that $u = 0$ and $v = 0$ has only the solution p.)

Consider the curve D on U defined by $f^* v = 0$. We have $f^* u = x_1 f^* v$. Thus $f^* u = 0$ on D as well, and $D = f^{-1}(p)$. □

Lemma 2. Let $\phi : S \dashrightarrow S'$ be a birational *rational* map of surfaces such that ϕ^{-1} is undefined at a point $p \in S'$. Then there is a curve C on S such that $\phi(C) = \{p\}$.

If ϕ were a morphism, this would follow immediately from the previous lemma.

Proof. Take the graph of the morphism

$$\begin{tikzcd} & S_1 \arrow[dl, "q"'] \arrow[dr, "q'"] & \\ S \arrow[rr, dashed, "\phi"] & & S' \end{tikzcd}$$

S_1 is the closure in $S \times S'$ of points $(x, f(x))$. This is a surface, possibly singular. Since ϕ^{-1} is undefined at $p \in S'$, ${q'}^{-1}$ is undefined at p as well. Then by the previous lemma, there is an irreducible curve $C_1 \subset S_1$ with $q'(C_1) = \{p\}$. Then the image of C_1 in S must be one-dimensional (as it sits inside $S \times S'$, and its image in S' is zero-dimensional). $\square$

Now we're ready to prove the Theorem, in 4 steps.

Proof. **Step 1: Set some notation.** Suppose the statement is false. Then the rational map $g : X \to \mathrm{Bl}_p(S)$ is not a morphism, so there is a $q \in X$ where g is undefined. (*Warning:* the diagonal arrows on the left should be dashed, but I don't know how to make them in TeX!)

$$\begin{tikzcd} & & \mathrm{Bl}_p(S) \arrow[dl, "h"] \arrow[dr, "\pi"] & & \\ \underset{\text{where } g \text{ not defined}}{q \in} & X \arrow[ur, "g \text{ (morphism?)}"] \arrow[rr, "f"] & & S & \underset{\text{where } f^{-1} \text{ not defined}}{\ni p} \end{tikzcd}$$

Note that g is defined away from $f^{-1}(p)$ (essentially because h and f are the same there: blowing up doesn't interfere with any of the surface besides p/E), so $f(q) = q$.

Step 2. By Lemma 2 applied to h, there is a curve $C \subset \mathrm{Bl}_p(S)$ such that $h(C) = q$. (Recall that this means that the rational map h is defined at all but finitely many points of C — call these the *good* points of E — and that h sends all of these good points to q.) Now $\pi(C) = f(h(C)) = f(q) = p$, so C is contained in the exceptional divisor. Thus C *is* the exceptional divisor, so $C = E$. Our diagram is now:

$$\begin{tikzcd} & & \mathrm{Bl}_p(S) \supset E \quad (h(E) = q) \arrow[dl, "h"] \arrow[dr, "\pi"] & & \\ \underset{\text{where } g \text{ not defined}}{q \in} & X \arrow[ur, "g \text{ (morphism?)}"] \arrow[rr, "f"] & & S & \underset{\text{where } f^{-1} \text{ not defined}}{\ni p} \end{tikzcd}$$

Step 3. Suppose y is a local coordinate for p. In the language of local rings, $y \in \mathfrak{m}_{p,S}$ but $y \notin \mathfrak{m}_{p,S}^2$.

Note that $\pi^* y$ is a local coordinate for all but finitely one point of E. (Draw a picture to explain why this is so; one can also use the patches of the blow-up.)

Note that $f^* y \in \mathfrak{m}_{q,X}$ (as a function on S vanishing at p pulls back by f to a function vanishing at q). But $f^* y \notin \mathfrak{m}_{q,X}^2$. Reason: otherwise, at all the good points of E, i.e. all but finitely many points of E, $\pi^* y = h^* f^* y$ which has multiplicity 2, i.e. $\pi^* y = h^* f^* y \in h^* \mathfrak{m}_{q,X}^2 \subset \mathfrak{m}^2_{\text{good point},\mathrm{Bl}_p(S)}$. But we said y was a local coordinate of al but one point of E.

Conclusion: For any lcoal coordinate for p, $f^*y \in \mathfrak{m}_{q,X} - \mathfrak{m}_{q,X}^2$.

(We're essentially done here in the complex-analytic case. We have a map from q to p and local coordinates at q which map to local coordinates at p. Hence the map is invertible. The last step makes this a bit more formal.)

Step 4. Suppose r is a good point of E, i.e. where $h : \mathrm{Bl}_p S \dashrightarrow X$ is defined. Choose local coordinates x, y near p so that a neighborhood of r is described as follows. The neighborhood of r in $\mathrm{Bl}_p S$ is given by $((x_0, y_0), [1; t])$ with the relation $y_0 = x_0 t$. x_0 and t are local coordinates of r in $\mathrm{Bl}_p S$, so in particular $t \in \mathfrak{m}_{r,\mathrm{Bl}_p(S)}$. $\pi^* x = x_0$ and $\pi^* y = y_0$.

By Lemma 1, there is a curve D through q in X such that $f(D) = p$. Say D is cut out by the local equation $z = 0$. Then $z \in \mathfrak{m}_{q,X}$ (as z vanishes at q). Then $f^* x = \alpha z$ and $f^* y = \beta z$ for some $\alpha, \beta \in \mathcal{O}_{q,X}$. As $f^* x, f^* y \notin \mathfrak{m}_{q,X}^2$, we must have $\alpha, \beta \notin \mathfrak{m}_{q,X}$. Thus the local function $f^*(y/x) \notin \mathfrak{m}_{q,X}$. Then $t = h^* f^*(y/x) \notin \mathfrak{m}_{r,\mathrm{Bl}_p(S)}$, contradiction. □

2. More applications of our theorems

If you want to classify surfaces up to birational invariance, you need only classify surfaces which *aren't* blow-ups, and then see which two are birationally the same. **Definition.** These are called minimal surfaces.

Explicitly: **Proposition.** Any surface can be obtained by repeatedly blowing up a minimal surface.

Proof. If it isn't minimal, blow down (decreasing the rank of the Neron-Severi group). Keep doing this until you can't blow down any more. □

Example: Any surface with no rational curves is minimal (e.g. an abelian surface). More generally, a surface with no rational curves of self-intersection -1 (i.e. no (-1)-curve) is minimal.

The follow key theorem tells you the converse:

Theorem (Castelnuovo's contractibility criterion). Let S be a surface and $E \subset S$ a rational curve with $E^2 = -1$.

Discussion in a moment.

Example. $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are both minimal. (**Exercise.** Check!) Thus we have a birational equivalence class (that of rational surfaces) with more than one minimal member. In fact, there are other minimal members of this class (Hirzebruch surfaces).

About Castelnuovo's criterion. I won't prove this in its entirety, but will say the strategy. We'll produce a line bundle that will give a map to projective space that will preserve $S - E$, but collapse E to a point. Then the harder part is to show that the point is a smooth point of the new surface. I'll omit that: the proofs involve some infinitesimal analysis beyond what I claimed are the prerequisites of the course.

Next day. I'll describe the construction of Castelnuovo's morphism.

Then we'll have amassed the basic tools to talk about surfaces. We'll then discuss ruled surfaces, and then rational surfaces!