# COMPLEX ALGEBRAIC SURFACES CLASS 7

RAVI VAKIL

## CONTENTS

**Recap of last time.** Last time we began discussing *blow-ups*:

Given $p \in S$, there is a surface $S' = Bl_p S$ and a morphism $\pi : S' \to S$, unique up to isomorphism, such that (i) the restriction of $\pi$ to $\pi^{-1}(S - \{p\})$ is an isomorphism onto $S - \{p\}$, and (ii) $\pi^{-1}(p)$ is isomorphic to $\mathbb{P}^1$. $\pi^{-1}(p)$ is called the exceptional divisor $p$, and is called the *exceptional divisor*.

A key example, and indeed the analytic-, formal-, or etale-local situation, was given by blowing up $S = \mathbb{A}^2$ at the origin, which I'll describe again soon when it comes up in a proof.

For the definition, complex analytically, you can take the same construction. Then you need to think a little bit about uniqueness. There is a more intrinsic definition that works algebraically, let $\mathcal{I}$ be the ideal sheaf of the point. Then $S' = \mathrm{Proj} \oplus_{d \geq 0} \mathcal{I}^d$.

## 1. HOW BASIC ASPECTS OF SURFACES CHANGE UNDER BLOW-UP

**Definition.** If $C$ is a curve on $S$, define the *strict transform* $C^{\mathrm{strict}}$ of $C$ to the the closure of the pullback on $S - p$, i.e. $\overline{\pi|_{S'-E}^*(C \cap S - p)}$. The *proper transform* $C^{\mathrm{proper}}$ is given by the pullback of the defining equation, so for example $\pi^* \mathcal{O}_S(C) = \mathcal{O}_{S'}(C')$.

**Lemma.** If $C$ has multiplicity $m$ at $p$, then $C^{\mathrm{proper}} = C^{\mathrm{strict}} + mE$, i.e. $\pi^* C \cong C^{\mathrm{strict}} + mE$.

*Proof.* The multiplicity of $C$ being $m$ means that in local coordinates, the defining equation has terms of degree $m$, but not lower. (Better: the defining equation lies in $\mathfrak{m}^m$ but not $\mathfrak{m}^{m+1}$.) Analytically, this means that the leading term in $x$ and $y$ has degree m.

*Date*: Wednesday, October 23.

We do this by local calculation which will be useful in general. (Draw picture.) $U_0 = \{((x_0, y_0), [1; v]) : y_0 = x_0 v\} = \operatorname{Spec} k[x_0, y_0, v]/y_0 = x_0 v = \operatorname{Spec} k[x_0, v]$. The exceptional divisor $E$ is given by $y_0 = 0$ (after morphism).

$U_1 = \{((x_1, y_1), [u; 1]) : x_1 = y_1 u\} = \operatorname{Spec} k[x_1, y_1, u]/x_1 = y_1 u = \operatorname{Spec} k[y_1, u]$. The exceptional divisor $E$ is given by $y_1 = 0$.

Map down to $(x, y) = \operatorname{Spec} k[x, y]$. $(x_0, y_0, v) \mapsto (x_0, y_0)$, $(x_1, y_1, v) \mapsto (x_1, y_1)$.

Given a function $f(x, y) = 0$. Pull it back to $U_0$: $f(x, y) = f_m(x, y) + \text{higher} = f_m(x_0, x_0 v) + \text{higher} + \cdots$. $\qquad\square$

**Exercise.** To see if you understood that, do the same calculation on patch 2.

**Theorem.** Suppose $\pi : S' \to S$ is a blow up of $S$ at $p$, with exceptional curve $E \subset S'$. Let $D$ and $D'$ be divisors on $S$. Then $\pi^* D \cdot \pi^* D' = D \cdot D'$, $E \cdot \pi^* D = 0$, $E^2 = -1$.

*Remark.* A curve on a smooth surface that is isomorphic to $\mathbb{P}^1$ and has self-intersection $-1$ is called a $(-1)$-*curve.*

*Proof.* The first we did yesterday. The second: by Serre's moving lemma, we can move $D$ away from $p$, then pull back. For the third: choose a curve $C$ passing through $p$ with multiplicity 1. (How to do this: hyperplane section of $S$.) Then $C^{\text{strict}} \cdot E = 1$. Also $C^{\text{proper}} \cdot E = 0$. as $C^{\text{strict}} + E = C^{\text{proper}}$, we're done.

**Theorem.** (a) There is an isomorphism $\operatorname{Pic} S \oplus \mathbb{Z} \xrightarrow{\sim} \operatorname{Pic} S'$ defined by $(D, n) \mapsto \pi^* D + nE$. (b) The same with $\operatorname{Pic}$ replaced by $NS$.

*Proof.* The arguments are the same for both parts, so I'll do (a). It is surjective: the divisors upstairs are either $E$ or strict transforms (which are proper transforms plus $E$'s). It is injective: if $\pi^* D + nE = 0$, then intersect with $E$ to see that $n = 0$; then apply $\pi_*$ to see that $D = 0$.

**Theorem.** $K_{S'} = \pi^* K_S + E$.

*Proof.* Clearly $K_{S'} = \pi^* K_S + mE$ for some $m$. By the adjunction formula for $E$, $K_E = K_{S'}(E)|_E$. Taking degrees:

$$-2 = (\pi^* K_S + mE + E) \cdot E = -m - 1.$$

$\qquad\square$

**Exercise/Remark.** If you want practice with the canonical bundle in local coordinates, take a meromorphic section of $K_S$ that has neither zero nor pole at $p$ (possible by Serre's moving lemma), write it as $f(x, y) dx \wedge dy$, and pull it back to the open set $U_1$ to see that you get $f(x_0, x_0 v) dx_0 \wedge d(x_0 v) = f(x_0, x_0 v) x_0 dx_0 \wedge dv$.

## 2. Rational maps of surfaces, linear systems, and elimination of indeterminacy

A rational map $S \dashrightarrow X$, where $X$ is a variety, means a morphism from an dense open set of $S$. Recall that a rational map from a curve $C$ to a projective variety can always be extended to a morphism. Similarly, a rational map from a surface $S$ to a projective variety can be extended over most points; the *set of indeterminacy* is a finite set of points. More precisely, given a map $\pi : S \dashrightarrow \mathbb{P}^n$. This is given by $n+1$ sections of some line bundle. It makes sense except where the sections are all zero. This will be in codimension 2.

Let $F$ be this finite set. We'll denote $\overline{\pi(S-F)}$ the image of $S$, and denote it $\pi(S)$. (I'm not sure we need to take the closure.) If $C$ is a curve *on* $S$, then we'll denote $\overline{\pi(C-F)}$ the image of $C$, and denote it $\pi(C)$. Here we definitely need to take the closure.

Now suppose you have a divisor $D$ on $S$. Given a subspace $V$ of dimension $n$ of $H^0(S, \mathcal{O}(D))$, we might hope to get a map to projective space $\mathbb{P}V^*$. (This is called a *linear system of dimension* $n$; I should have introduced this notation earlier.) If it is base point free, we do.

If it has base points, the locus could have components of dimension 1. Such a component is called a *fixed component* of the linear system $V$. The *fixed part* of $V$ is the biggest divisor contained in every element of $V$. So if this fixed part is $F$, then $D - F$ has no fixed components.

(I'm not happy with how I explained the previous paragraphs in class. I hope this is clearer.)

**Lemma.** If the linear systems has no fixed part, then it has only a finite number of fixed points.

*Proof.* Take two general sections, and look at their two zero-sets. Where do they intersect? At a bunch of points. Hence we get at most $D^2$? $\square$

We've basically shown that there is a bijection between:

(i) { rational maps $\pi : S \dashrightarrow \mathbb{P}^n$ such that $\pi(S)$ is contained in no hyperplane }

(ii) { linear systems on $S$ without fixed part and of dimension $n$ }

(Explain the correspondence.)

**Theorem (Elimination of indeterminacy).** Let $\pi : S \dashrightarrow X$ be a rational map from a surface to a projective variety. Then there exists a surface $S'$, a morphism $\eta : S' \to S$ which is the composite of a finite number of blow-ups, and a morphism $f : S' \to X$ such

that the diagram

$$S'$$
$$\swarrow{\scriptstyle \eta} \qquad \searrow{\scriptstyle f}$$
$$S \quad \overset{\pi}{\dashrightarrow} \quad X$$

is commutative.

*Proof.* Idea: blow up fixed points, show that $D^2$ decreases.

We immediately reduce to the case where $X$ is $\mathbb{P}^m$, and $\pi(S)$ isn't contained in any hyperplane of $\mathbb{P}^m$. Then $\phi$ corresponds to a linear system $V \subset |D|$ of dimension $n$ on $S$, with no fixed component. If $V$ has no base point, then we're done.

Otherwise, we blow up a base point $x$, and consider $S_1 \to S$ at $x$ (and hence a rational map $S_1 \dashrightarrow S$). The exceptional curve is now in the fixed part of the linear system, with some multiplicity $k \geq 1$. So we subtract $kE$ to get rid of the fixed part, i.e. get a new linear system $V_1 \subset |\pi^* D - kE|$, to get the same rational map $\phi_1 : S_1 \dashrightarrow S$, given by $D_1 = D - kE$. If this is a morphism, we win, otherwise we keep going.

At some point, this process must stop (and hence we win in the long run). We prove this is the case when $D^2 = i$, by induction on $i$. Base case, $i = 0$: the number of fixed points is bounded by $D^2 = 0$, so there aren't any. Inductive step: Now $i > 0$. Then we blow-up once, and we get a new surface with divisor class. On this surface, $D_1^2 = (D - kE)(D - kE) = D^2 - k^2 < D^2$. So by the inductive hypothesis, the process will terminate on this new surface, completing the induction. $\square$

## 3. THE UNIVERSAL PROPERTY OF BLOWING UP

**Theorem (Universal property of blowing up).** Let $f : X \to S$ be a birational morphism of surfaces, and suppose that the rational map $f^{-1}$ is undefined at a point $p$ of $S$. Then $f$ factorizes as
$$f : X \overset{g}{\to} \tilde{S} := \mathrm{Bl}_p S \overset{\pi}{\to} S$$
where $g$ is a birational morphism and $\pi$ is the blow-up at $p$.

Proof: next day.

### 3.1. **Applications of the universal property of blowing up.** Two theorems.

**Theorem (all birational morphisms factor into blow-ups).** Let $f : S \to S_0$ be a birational morphism of surfaces. Then there is a sequence of blow-ups $\pi_k : S_k \to S_{k-1}$ $(k-1, \ldots, n)$ and an isomorphism $u : S \overset{\sim}{\to} S_n$ such that $f = \pi_1 \circ \cdots \circ \pi_n \circ u$.

*Proof.* If $f$ is an isomorphism, we're done. Otherwise, there is a point $p$ of $S_0$ such that $f^{-1}$ is undefined At $p$, and we can factor through $S \to S_1 = \mathrm{Bl}_p S_0$. We can repeat this.

If $n(f_k)$ is the number of contracted curves of $n(f_k) < n(f_{k-1})$: if $E$ is the exceptional divisor of $\pi_k : S_k \to S_{k-1}$, then the preimage of $E$ in $S$ contains a curve which is contracted

by $f_{k-1}$ but not $f_k$. As the number of contracted curves can't be negative, the process must terminate. $\quad\square$

**Theorem (all birational maps can be factored into blow-ups).** Let $\phi : S \dashrightarrow S'$ be a birational map of surfaces. Then there is a surface $S''$ and a commutative diagram

$$
\begin{array}{ccc}
 & S'' & \\
{\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\
S & \overset{\phi}{\dashrightarrow} & S'
\end{array}
$$

where the morphisms $f$ and $g$ are composites of blow-ups.

*Proof.* By the theorem of elimination of indeterminacy, we can find such a diagram such that $f$ is a composition of blow-ups. By the Theorem above, $g$ must then be a composition of blow-ups too. $\quad\square$

We've now proved some powerful stuff, so let's take a step back and see what we now know, and how it relates to classification.

Two surfaces are birational iff they can be be related by sequences of blow-ups. We'll be interested in birational classification, but biregular classification is very close.

If $f : S \to S'$ is birational which is the composition of $n$ blow-ups, then $NS(S) \cong NS(S') \oplus \mathbb{Z}^n$, so $n$ is independent of the choice of blow-ups. **Exercise:** Use this to show that every birational morphism from $S$ to itself is an isomorphism.

**Fact.** In a blow-up, $H^i$ of the structure sheaf is preserved, i.e. if $\pi : S' \to S$ is a blow-up, then $\pi^* : H^i(\mathcal{O}_S) \to H^i(\mathcal{O}_{S'})$ is an isomorphism.

The algebraic way of proving this fact comes from the Leray spectral sequence, and the fact that $\pi_*\mathcal{O}_{S'} = \mathcal{O}_S$ and $R^i\pi_*\mathcal{O}_{S'} = 0$ for $i > 0$. This in turn requires some infinitesimal analysis, in the form of "formal function theorems". I suspect that there should be a relatively straightforward analytic proof.

In particular, by these numbers are birational invariants.

So look at what this means for the Hodge diamond. When you blow up, you add 1 to the central entry (the rank of the Neron-Severi group). Everything else is constant.

**Next day:** More consequences of these powerful theorems. Proof of the universal property of blowing up. Castelnuovo's criterion for blowing down curves.