COMPLEX ALGEBRAIC SURFACES CLASS 7

RAVI VAKIL

CONTENTS

Recap of last time. Last time we began discussing *blow-ups*:

Given $p \in S$, there is a surface $S' = Bl_p S$ and a morphism $\pi : S' \to S$, unique up to isomorphism, such that (i) the restriction of π to $\pi^{-1}(S - \{p\})$ is an isomorphism onto $S - \{p\}$, and (ii) $\pi^{-1}(p)$ is isomorphic to $\mathbb{P}^1$. $\pi^{-1}(p)$ is called the exceptional divisor p, and is called the *exceptional divisor*.

A key example, and indeed the analytic-, formal-, or etale-local situation, was given by blowing up $S = \mathbb{A}^2$ at the origin, which I'll describe again soon when it comes up in a proof.

For the definition, complex analytically, you can take the same construction. Then you need to think a little bit about uniqueness. There is a more intrinsic definition that works algebraically, let $\mathcal{I}$ be the ideal sheaf of the point. Then $S' = \operatorname{Proj} \oplus_{d \geq 0} \mathcal{I}^d$.

1. HOW BASIC ASPECTS OF SURFACES CHANGE UNDER BLOW-UP

Definition. If C is a curve on S, define the *strict transform* C^{strict} of C to the the closure of the pullback on $S - p$, i.e. $\overline{\pi|^*_{S'-E}(C \cap S - p)}$. The *proper transform* C^{proper} is given by the pullback of the defining equation, so for example $\pi^* \mathcal{O}_S(C) = \mathcal{O}_{S'}(C')$.

Lemma. If C has multiplicity m at p, then $C^{\text{proper}} = C^{\text{strict}} + mE$, i.e. $\pi^* C \cong C^{\text{strict}} + mE$.

Proof. The multiplicity of C being m means that in local coordinates, the defining equation has terms of degree m, but not lower. (Better: the defining equation lies in $\mathfrak{m}^m$ but not $\mathfrak{m}^{m+1}$.) Analytically, this means that the leading term in x and y has degree m.

Date: Wednesday, October 23.

We do this by local calculation which will be useful in general. (Draw picture.) $U_0 = \{((x_0, y_0), [1; v]) : y_0 = x_0 v\} = \operatorname{Spec} k[x_0, y_0, v]/y_0 = x_0 v = \operatorname{Spec} k[x_0, v]$. The exceptional divisor E is given by $y_0 = 0$ (after morphism).

$U_1 = \{((x_1, y_1), [u; 1]) : x_1 = y_1 u\} = \operatorname{Spec} k[x_1, y_1, u]/x_1 = y_1 u = \operatorname{Spec} k[y_1, u]$. The exceptional divisor E is given by $y_1 = 0$.

Map down to $(x, y) = \operatorname{Spec} k[x, y]$. $(x_0, y_0, v) \mapsto (x_0, y_0)$, $(x_1, y_1, v) \mapsto (x_1, y_1)$.

Given a function $f(x, y) = 0$. Pull it back to U_0: $f(x, y) = f_m(x, y) + \text{higher} = \mathrm{f_m(x_0, x_0 v)} + \text{higher} + \cdots$. □

Exercise. To see if you understood that, do the same calculation on patch 2.

Theorem. Suppose $\pi : S' \to S$ is a blow up of S at p, with exceptional curve $E \subset S'$. Let D and D' be divisors on S. Then $\pi^* D \cdot \pi^* D' = D \cdot D'$, $E \cdot \pi^* D = 0$, $E^2 = -1$.

Remark. A curve on a smooth surface that is isomorphic to $\mathbb{P}^1$ and has self-intersection -1 is called a (-1)-*curve*.

Proof. The first we did yesterday. The second: by Serre's moving lemma, we can move D away from p, then pull back. For the third: choose a curve C passing through p with multiplicity 1. (How to do this: hyperplane section of S.) Then $C^{\text{strict}} \cdot E = 1$. Also $C^{\text{proper}} \cdot E = 0$. as $C^{\text{strict}} + E = C^{\text{proper}}$, we're done.

Theorem. (a) There is an isomorphism $\operatorname{Pic} S \oplus \mathbb{Z} \xrightarrow{\sim} \operatorname{Pic} S'$ defined by $(D, n) \mapsto \pi^* D + nE$. (b) The same with Pic replaced by NS.

Proof. The arguments are the same for both parts, so I'll do (a). It is surjective: the divisors upstairs are either E or strict transforms (which are proper transforms plus E's). It is injective: if $\pi^* D + nE = 0$, then intersect with E to see that $n = 0$; then apply π_* to see that $D = 0$.

Theorem. $K_{S'} = \pi^* K_S + E$.

Proof. Clearly $K_{S'} = \pi^* K_S + mE$ for some m. By the adjunction formula for E, $K_E = K_{S'}(E)|_E$. Taking degrees:

$$-2 = (\pi^* K_S + mE + E) \cdot E = -m - 1.$$

□

Exercise/Remark. If you want practice with the canonical bundle in local coordinates, take a meromorphic section of K_S that has neither zero nor pole at p (possible by Serre's moving lemma), write it as $f(x, y) dx \wedge dy$, and pull it back to the open set U_1 to see that you get $f(x_0, x_0 v) dx_0 \wedge d(x_0 v) = f(x_0, x_0 v) x_0 dx_0 \wedge dv$.

2. Rational maps of surfaces, linear systems, and elimination of indeterminacy

A rational map $S \dashrightarrow X$, where X is a variety, means a morphism from an dense open set of S. Recall that a rational map from a curve C to a projective variety can always be extended to a morphism. Similarly, a rational map from a surface S to a projective variety can be extended over most points; the *set of indeterminacy* is a finite set of points. More precisely, given a map $\pi : S \dashrightarrow \mathbb{P}^n$. This is given by $n+1$ sections of some line bundle. It makes sense except where the sections are all zero. This will be in codimension 2.

Let F be this finite set. We'll denote $\overline{\pi(S-F)}$ the image of S, and denote it $\pi(S)$. (I'm not sure we need to take the closure.) If C is a curve *on* S, then we'll denote $\overline{\pi(C-F)}$ the image of C, and denote it $\pi(C)$. Here we definitely need to take the closure.

Now suppose you have a divisor D on S. Given a subspace V of dimension n of $H^0(S, \mathcal{O}(D))$, we might hope to get a map to projective space $\mathbb{P}V^*$. (This is called a *linear system of dimension* n; I should have introduced this notation earlier.) If it is base point free, we do.

If it has base points, the locus could have components of dimension 1. Such a component is called a *fixed component* of the linear system V. The *fixed part* of V is the biggest divisor contained in every element of V. So if this fixed part is F, then $D-F$ has no fixed components.

(I'm not happy with how I explained the previous paragraphs in class. I hope this is clearer.)

Lemma. If the linear systems has no fixed part, then it has only a finite number of fixed points.

Proof. Take two general sections, and look at their two zero-sets. Where do they intersect? At a bunch of points. Hence we get at most D^2? ☐

We've basically shown that there is a bijection between:

(i) { rational maps $\pi : S \dashrightarrow \mathbb{P}^n$ such that $\pi(S)$ is contained in no hyperplane }

(ii) { linear systems on S without fixed part and of dimension n }

(Explain the correspondence.)

Theorem (Elimination of indeterminacy). Let $\pi : S \dashrightarrow X$ be a rational map from a surface to a projective variety. Then there exists a surface S', a morphism $\eta : S' \to S$ which is the composite of a finite number of blow-ups, and a morphism $f : S' \to X$ such that the diagram

$$\begin{tikzcd} & S' \arrow[dl, "\eta"'] \arrow[dr, "f"] & \\ S \arrow[rr, dashed, "\pi"] & & X \end{tikzcd}$$

is commutative.

Proof. Idea: blow up fixed points, show that D^2 decreases.

We immediately reduce to the case where X is $\mathbb{P}^m$, and $\pi(S)$ isn't contained in any hyperplane of $\mathbb{P}^m$. Then ϕ corresponds to a linear system $V \subset |D|$ of dimension n on S, with no fixed component. If V has no base point, then we're done.

Otherwise, we blow up a base point x, and consider $S_1 \to S$ at x (and hence a rational map $S_1 \dashrightarrow S$). The exceptional curve is now in the fixed part of the linear system, with some multiplicity $k \geq 1$. So we subtract kE to get rid of the fixed part, i.e. get a new linear system $V_1 \subset |\pi^* D - kE|$, to get the same rational map $\phi_1 : S_1 \dashrightarrow S$, given by $D_1 = D - kE$. If this is a morphism, we win, otherwise we keep going.

At some point, this process must stop (and hence we win in the long run). We prove this is the case when $D^2 = i$, by induction on i. Base case, $i = 0$: the number of fixed points is bounded by $D^2 = 0$, so there aren't any. Inductive step: Now $i > 0$. Then we blow-up once, and we get a new surface with divisor class. On this surface, $D_1^2 = (D - kE)(D - kE) = D^2 - k^2 < D^2$. So by the inductive hypothesis, the process will terminate on this new surface, completing the induction. $\square$

3. The universal property of blowing up

Theorem (Universal property of blowing up). Let $f : X \to S$ be a birational morphism of surfaces, and suppose that the rational map f^{-1} is undefined at a point p of S. Then f factorizes as

$$f : X \xrightarrow{g} \tilde{S} := \mathrm{Bl}_p S \xrightarrow{\pi} S$$

where g is a birational morphism and π is the blow-up at p.

Proof: next day.

3.1. Applications of the universal property of blowing up.

Two theorems.

Theorem (all birational morphisms factor into blow-ups). Let $f : S \to S_0$ be a birational morphism of surfaces. Then there is a sequence of blow-ups $\pi_k : S_k \to S_{k-1}$ $(k - 1, \dots, n)$ and an isomorphism $u : S \xrightarrow{\sim} S_n$ such that $f = \pi_1 \circ \cdots \circ \pi_n \circ u$.

Proof. If f is an isomorphism, we're done. Otherwise, there is a point p of S_0 such that f^{-1} is undefined At p, and we can factor through $S \to S_1 = \mathrm{Bl}_p S_0$. We can repeat this.

If $n(f_k)$ is the number of contracted curves of $n(f_k) < n(f_{k-1})$: if E is the exceptional divisor of $\pi_k : S_k \to S_{k-1}$, then the preimage of E in S contains a curve which is contracted by f_{k-1} but not f_k. As the number of contracted curves can't be negative, the process must terminate. $\square$

Theorem (all birational maps can be factored into blow-ups). Let $\phi : S \dashrightarrow S'$ be a birational map of surfaces. Then there is a surface S'' and a commutative diagram

$$\begin{tikzcd} & S'' \arrow[dl, "f"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "\phi"] & & S' \end{tikzcd}$$

where the morphisms f and g are composites of blow-ups.

Proof. By the theorem of elimination of indeterminacy, we can find such a diagram such that f is a composition of blow-ups. By the Theorem above, g must then be a composition of blow-ups too. $\square$

We've now proved some powerful stuff, so let's take a step back and see what we now know, and how it relates to classification.

Two surfaces are birational iff they can be be related by sequences of blow-ups. We'll be interested in birational classification, but biregular classification is very close.

If $f : S \to S'$ is birational which is the composition of n blow-ups, then $NS(S) \cong NS(S') \oplus \mathbb{Z}^n$, so n is independent of the choice of blow-ups. **Exercise:** Use this to show that every birational morphism from S to itself is an isomorphism.

Fact. In a blow-up, H^i of the structure sheaf is preserved, i.e. if $\pi : S' \to S$ is a blow-up, then $\pi^* : H^i(\mathcal{O}_S) \to H^i(\mathcal{O}_{S'})$ is an isomorphism.

The algebraic way of proving this fact comes from the Leray spectral sequence, and the fact that $\pi_* \mathcal{O}_{S'} = \mathcal{O}_S$ and $R^i \pi_* \mathcal{O}_{S'} = 0$ for $i > 0$. This in turn requires some infinitesimal analysis, in the form of "formal function theorems". I suspect that there should be a relatively straightforward analytic proof.

In particular, by these numbers are birational invariants.

So look at what this means for the Hodge diamond. When you blow up, you add 1 to the central entry (the rank of the Neron-Severi group). Everything else is constant.

Next day: More consequences of these powerful theorems. Proof of the universal property of blowing up. Castelnuovo's criterion for blowing down curves.