COMPLEX ALGEBRAIC SURFACES CLASS 6

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CONTENTS

I realize now that I am starting to be sloppy about notation for divisors and their corresponding invertible sheaves. For example, roman characters usually refer to divisors, e.g. D, and calligraphic characters usually refer to sheaves. The main cause of confusion is that we use additive notation for divisors, and multiplicative notation for sheaves. I'll likely continue this confusion, so please bear with me. Divisors will normally be considered up to linear equivalence, so it makes sense to talk about a divisor K in the class of $\mathcal{K}$.

Recap of last time. We began intersection theory on a surface S. We constructed $\operatorname{Div} S \times \operatorname{Div} S \to Z$ which in fact descended to $\operatorname{Pic} S \times \operatorname{Pic} S \to Z$.

In the case of two curves with no common components, we had a local intersection number: if the local equations at $x \in S$ of the two curves are given by f and g,

$$m_x(C \cap C') = \dim_k \mathcal{O}_x/(f, g).$$

The intersection number $C \cdot C'$ is defined by

$$C \cdot C' = \sum_{x \in C \cap C'} m_x(C \cap C').$$

We did some examples. **Exercise.** (a) Calculate the local intersection number (at $(0,0)$) of $y = 0$ and $y = x^n$. (b) $y = x^n$ and $y^2 = x^2$, where $n > 1$. (c) Verify that the local intersection number is 1 if $f = 0$ and $g = 0$ are smooth at $(0,0)$ and have distinct tangent directions.

For $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic} S$, we defined define

$$\mathcal{L} \cdot \mathcal{L}' = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^*) - \chi(\mathcal{L}'^*) + \chi(\mathcal{L}^* \otimes \mathcal{L}'^*).$$

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Theorem. If C and C' are two distinct irreducible curves on S then $\mathcal{O}_S(C)\cdot\mathcal{O}_S(C') = C\cdot C'$.

Proof was by exactness of

$$0 \to \mathcal{O}_S(-C-C') \stackrel{(s',-s)}{\to} \mathcal{O}_S(-C)\oplus\mathcal{O}_S(-C') \stackrel{(s,s')}{\to} \mathcal{O}_S \to \mathcal{O}_{C\cap C'} \to 0$$

and taking Euler characteristics. Here $C\cap C'$ is taken scheme-theoretically.

I was fuzzier than I would have liked in describing these morphisms. (Describe them.)

Proposition. If C is a non-singular irreducible curve on S, and $\mathcal{L}\in \operatorname{Pic} S$, then $\mathcal{O}_S(C)\cdot\mathcal{L} = \deg \mathcal{L}|_C$. For example, $C\cdot C$ is the degree of the normal bundle.

We'll use this in a minute.

1. The intersection form

We'll use the result of Serre that I mentioned before: if D is any divisor, and H is very ample, then then $D+nH$ is very ample for sufficiently large n. (Proof available upon request.) In particular, D can be written as $(D+nH)-(nH)$, the difference of two smooth curves on S. (Here we use Bertini's theorem.) You should think of this as a moving lemma.

Theorem. $\cdot$ is symmetric bilinear.

Proof. Symmetry is obvious. Now let's prove bilinearity. Consider $s(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3) := (\mathcal{L}_1\cdot(\mathcal{L}_2\otimes\mathcal{L}_3)) - (\mathcal{L}_1\cdot\mathcal{L}_2) - (\mathcal{L}_1\cdot\mathcal{L}_3)$. (i) This is 0 when $\mathcal{L}_1$ is the class of a smooth curve, by our proposition.

(ii) This is symmetric in $\mathcal{L}_i$: plug into the formula.

(iii) Next, suppose $\mathcal{L}$ and $\mathcal{L}'$ are any two invertible sheaves. By Serre's result I mentioned last day, $\mathcal{L}' = \mathcal{O}(A-B)$ where A and B are two smooth curves. $s(\mathcal{L},\mathcal{L}',\mathcal{O}_S(B)) = 0$ by (i). Expanding, we get $0 = \mathcal{L}\cdot\mathcal{O}_S(A) - \mathcal{L}\cdot\mathcal{L}' - \mathcal{L}\cdot\mathcal{O}_S(B)$ from which

$$\mathcal{L}\cdot\mathcal{L}' = \mathcal{L}\cdot\mathcal{O}_S(A) - \mathcal{L}\cdot\mathcal{O}_S(B).$$

The right side is linear in $\mathcal{L}$, so the left side is too. $\square$

Examples. (i) If C is a smooth curve, and $f: S\to C$ is a surjective morphism, F is a fiber of f. Then $F^2=0$.

(Topologically, this is believable.) $F = f^*[x]$ for some $x\in C$. There is a divisor A on C, linearly equivalent to x, such that $x\notin A$, and $F\equiv f^*A$. Since f^*A is a linear combination of fibers of f all distinct from F, we have $F^2 = F\cdot f^*A = 0$.

(ii) Let S' be a surface, $g: S'\to S$ a generically finite morphism of degree d, and D and D' divisors on S. Then $g^*D\cdot g^*D' = dD\cdot D'$.

Using Serre's result, it suffices to prove the formula when D and D' are hyperplane sections of S. There's a big open set U of S' over which g is etale (i.e. local isomorphism over $\mathbb{C}$) Move D and D' so that they meet transversely, and their intersection lies in U. Then g^*D and g^*D' also meet transversely, and $g^*D \cap g^*D' = g^{-1}(D \cap D')$. (This is dodgy in positive characteristic, but the result is true.)

(iii) Example: $\mathbb{P}^2$. We get *Bezout's theorem.* A degree d curve meets a degree d' curve in dd' points, counted correctly.

1.1. **The Neron-Severi group.**

Now I want to go to the complex analytic topology. Consider

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0.$$

Look at the long exact sequence in cohomology. Strip off the H^0's to get

$$0 \to H^1(S, \mathbb{Z}) \to H^1(S, \mathcal{O}_S) \to \operatorname{Pic}(S) \to H^2(S, \mathbb{Z}) \to H^2(S, \mathcal{O}_S).$$

$0 \to T \to \operatorname{Pic} S \to NS(S) \to 0$. Hodge theory shows that T is a complex torus, denoted $\operatorname{Pic}^0(S)$; in fact this is an algebraic variety. The *Neron-Severi group* $NS(S)$ is a finitely generated group. This all holds true over an arbitrary field, but is quite hard.

The map $\operatorname{Pic} S \to H_2(S, \mathbb{Z})$ is what you think it is topologically. The bilinear form on $\operatorname{Pic} S$ turns into the intersection form $H^2 \times H^2 \to H^4 = \mathbb{Z}$.

1.2. **Aside: The Hodge diamond of a complex projective surface.**

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

$H^i(S, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}(S)$.

$H^{p,q}(S) = H^q(S, \wedge^p \Omega_S)$. (Often written Ω_S^p.)

Hence: put 1's ; irregularity $q = h^{1,1}$; genus $h^{2,0}$. $h^{0,0} = 1$ of course; $h^{2,2} = h^2(S, \mathcal{K}_S)$, which I said earlier as part of Serre duality.

Pairing in cohomology agrees with pairing of description in terms of forms. Duality holds.

Our map $\operatorname{Pic}(S) \to H^2(S, \mathbb{Z})$ actually lies in $H^{1,1}(S)$.

Lefschetz (1,1)-theorem. The Neron-Severi group, i.e. the image of Pic, is all of $H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$.

2. RIEMANN-ROCH FOR SURFACES

Riemann-Roch theorem for surfaces. For all $\mathcal{L} \in \operatorname{Pic} S$,

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_S) + \frac{1}{2}\mathcal{L} \cdot (\mathcal{L} - \mathcal{K}).$$

Proof.

$$\begin{aligned} \mathcal{L}^* \cdot (\mathcal{L} \otimes \mathcal{K}^*) &= \chi(\mathcal{O}_S) - \chi(\mathcal{L}) - \chi(\mathcal{K} \otimes \mathcal{L}^*) + \chi(\mathcal{K}_S) \text{ (by def'n)} \\ &= \chi(\mathcal{O}_S) - \chi(\mathcal{L}) - \chi(\mathcal{L}) + \chi(\mathcal{O}_S) \text{ (by Serre duality, twice)} \\ &= 2(\chi(\mathcal{O}_S) - \chi(\mathcal{L})) \end{aligned}$$

and we're done after rearranging. □

A typical way of using it is:

$$h^0(D) + h^0(K - D) - h^1(D) = \chi(\mathcal{O}_S) + \frac{D \cdot (D - K)}{2} \Rightarrow$$

$$h^0(D) + h^0(K - D) \geq \chi(\mathcal{O}_S) + \frac{D \cdot (D - K)}{2}.$$

Example: If H is any divisor, then

$$\chi(S, nH) = \frac{1}{2}(nH) \cdot (nH - K) + \chi(\mathcal{O}_S) = (H^2/2)n^2 + (-HK/2)n + (\chi(\mathcal{O}_S)),$$

so we have quadratic growth of the Euler characteristic. In particular, if H is ample, then for $n \gg 0$, $h^0(S, nH) = \cdots$.

Another useful consequence:

The genus formula. Let C be an irreducible curve on a surface S. The genus of C, defined by $g(C) = h^1(C, \mathcal{O}_C)$, is given by $g(C) = 1 + \frac{1}{2}(C^2 + C \cdot K)$.

Proof. Use $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$. Take Euler characteristics to get

$$1 - g(C) = \chi(\mathcal{O}_C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) = \cdots.$$

(**Exercise:** finish this.) But we already knew this, using the adjunction formula.

Proof using the adjunction formula. $\deg \mathcal{K}_C = \deg(K + C)|_C$.

Remark: Arithmetic genus of a singular curve.

The modern form of the Riemann-Roch form has a second part, proved long ago:

Noether's formula (fact). $\chi(\mathcal{O}_S) = \frac{1}{12}(K^2 + c_2(T_S)) = \frac{1}{12}(K^2 + \chi_{top}(S))$. The latter equality is only in the complex case.

3. Blow-ups: an example of a birational morphism

The key example of a birational morphism is *blowing up at a point*, also sometimes called a monoidal transformation. In fact, we'll see that all birational morphisms are series of monoidal transformations.

Fact. Given $p \in S$, there is a surface $S' = Bl_p S$ and a morphism $\pi : S' \to S$, unique up to isomorphism, such that (i) the restriction of π to $\pi^{-1}(S - \{p\})$ is an isomorphism onto $S - \{p\}$, and (ii) $\pi^{-1}(p)$ is isomorphic to $\mathbb{P}^1$.

$\pi^{-1}(p)$ is called the exceptional divisor p, and is called the *exceptional divisor*.

Example: blowing up $S = \mathbb{A}^2$ at the origin. $S' = \{(x, l) : l \text{ line through } (0,0), x \in l\}$. Clearly we have $\pi : S' \to S$, and S' is smooth (smooth choice of line, then smooth choice of point on line).

Algebraically: l is parametrized by $\mathbb{P}^1$, parametrize by $[u;1]$ and $[1;v]$. Patch 1: $[1;v] = [x_0;y_0]$, i.e. $y_0 = x_0 v$. The exceptional divisor is $x_0 = 0$. Patch 2: $[u;1] = [x_1;y_1]$, i.e. $x_0 = y_0 u$. The exceptional divisor is $y_0 = 0$.

Note that the map $\mathbb{A}^2 - (0,0) \to \mathbb{P}^1$ given by $(x,y) \mapsto [x;y]$ couldn't be extended over the origin. But it can be extended to $S' \to \mathbb{P}^1$; this blow-up "resolves the indeterminacy of the map". This is a feature, as we'll see.

Defining the blow-up in general. Complex analytically, you can take the same construction. Then you need to think a little bit about uniqueness.

There is a more intrinsic definition that works algebraically, let $\mathcal{I}$ be the ideal sheaf of the point. Then $S' = \operatorname{Proj} \oplus_{d \geq 0} \mathcal{I}^d$.

Definition. If C is a curve on S, define the *strict transform* C^{strict} of C to the the closure of the pullback on $S - p$, i.e. $\overline{\pi|^*_{S'-E}(C \cap S - p)}$. The proper transform C^{proper} is given by the pullback of the defining equation, so for example $\pi^* \mathcal{O}_S(C) = \mathcal{O}_{S'}(C')$.

Next day, we will investigate the relationship between S and S', and between curves on the two surfaces. In particular:

Lemma. If C has multiplicity m at p, then $C^{\text{proper}} = C^{\text{strict}} + mE$.

Theorem. (a) There is an isomorphism $\operatorname{Pic} S \oplus \mathbb{Z} \xrightarrow{\sim} \operatorname{Pic} S'$ defined by $(D, n) \mapsto \pi^* D + nE$.
(b) The same with Pic replaced by NS.

Theorem. $K_{S'} = \pi^* K_S + E$.

and more...