COMPLEX ALGEBRAIC SURFACES CLASS 5

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CONTENTS

Recap of last time. We mainly talked about two theorems:

Fact: Serre duality. If X is proper nonsingular and dimension n, then for $0 \leq i \leq n$,

$$H^i(X, \mathcal{L}) \otimes H^{n-i}(X, \mathcal{K} \otimes \mathcal{L}^*) \to H^n(X, \mathcal{K}) \sim \mathbb{C}$$

is a perfect pairing.

Fact: Riemann-Roch Theorem.

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg(\mathcal{L}) - g + 1.$$

We then discussed applications of Riemann-Roch. In particular:

Theorem.

(a) If $\deg \mathcal{L} \geq 2g-1$, then $h^1(C, \mathcal{L}) = 0$. Hence $h^0(C, \mathcal{L}) = \deg \mathcal{L} - g + 1$.
(b) If $\deg \mathcal{L} \geq 2g$, then $\mathcal{L}$ is
(c) If $\deg \mathcal{L} \geq 2g+1$, then $\mathcal{L}$ is very ample.

Finally: **The adjunction formula.** $\mathcal{K}_D = \mathcal{K}_X(D)|_D$.

There's one fact I should have mentioned. In one example, we applied this to find the genus of a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^3$, to find its genus. (It was 4.) How do you know that there *are* smooth complete intersections? Answer:

Bertini's theorem (fact). A general hyperplane section of a smooth variety is smooth. (Not hard, but proof omitted.)

Date: Wednesday, October 16.

1. Classification of curves of genus 3

Theorem. Every smooth curve of genus 3 is of (precisely) one of the two following forms.

(i) A smooth quartic curve in $\mathbb{P}^2$ (in a unique way, up to automorphisms of $\mathbb{P}^2$).

(ii) A double cover of $\mathbb{P}^1$ branched over 8 points (in a unique way, up to automorphisms of $\mathbb{P}^2$).

All the parts of this proof are straightforward, but there are many parts, so I won't go through much. If you're interested in Riemann surfaces, I can explain it to you. **(Exercise.)**

- Every smooth quartic is embedded by the canonical sheaf (use adjunction). So that's uniqueness.
- Next: if the canonical sheaf isn't very ample, we need to show that we're in case (ii).
- Every cover of $\mathbb{P}^1$ branched over 8 points is genus 3.
- In fact, case (ii) is the limit of case (i)
- The dimension of the moduli space is 6.
- Curves of genus 4: most are complete intersections of quadric and cubic in $\mathbb{P}^3$. Curves of genus 5: most are complete intersections of three quadrics in $\mathbb{P}^4$.

1.1. **Kodaira dimension.** The philosophy behind that last example carries over to surfaces. Someone hands you an arbitrary projective surface S, and asks you to classify it. You don't have much to go on. One idea is to take the canonical bundle, and do something with it, ideally map the surface to projective space. Even if K won't do it, some multiple might.

Hence for $n >> 0$, consider the linear system $|nK|$. Even if there are sections, they might have base points. But then at least we have a rational map to projective space $S \dashrightarrow \mathbb{P}^N$. We can define its image as its closure. The first coarse invariant is: what is the dimension of the image? This invariant is called *Kodaira dimension* κ. (This turns out to be equivalent to looking at the rate of growth of $h^0(S, \mathcal{K}^{\otimes n})$ as n gets large; $h^0(S, \mathcal{K}^{\otimes n}) = O(n^\kappa)$.)

If $\mathcal{K}$ never has sections, we don't even get a rational map to projective space, and by convention $\kappa = -\infty$. Clearly $\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}$ in general. If $\kappa(X) = \dim X$, then X is said to be of *general type*.

(Finer distinctions are achieved by looking at the linear system $|nK|$ for smaller K.)

Here's how this plays out for curves. Genus greater than 1: $\deg \mathcal{K} = 2g - 2$, so a small multiple of $\mathcal{K}$ will be very ample. Thus the image is dimension 1.

Genus 1: $\mathcal{K} \cong \mathcal{O}$, so any power of $\mathcal{K}$ is trivial, and we get only one section. Hence the complete linear system sends C to a point. $\kappa = 0$.

Genus 0: $\mathcal{K}$ has negative degree, so now (positive) tensor power of it has sections. Here we say $\kappa = -\infty$.

So Kodaira dimension divides curves into three groups: rational, elliptic, and general type. We can further divide general type into cases, for example according to $h^0(C, \mathcal{K})$, i.e. genus, or by $\deg \mathcal{K} = 2g - 2$.

In the surface case, we'll see a rough breakdown according to $\kappa = -\infty, 0, 1$, and 2.

On to surfaces!

2. Intersection theory on surfaces

S is now a surface. We want to construct an intersection form $\operatorname{Div} S \times \operatorname{Div} S \to \mathbb{Z}$.

First, let's make sense of what the intersection should be if it is proper.

Definition. Suppose C and C' are two curves with no common components on a surface S, $x \in C \cap C'$, $\mathcal{O}_x$ the local ring of S at x. If f (resp. g) is an equation of C (resp. C') in $\mathcal{O}_x$, the intersection multiplicity of C and C' at x is defined to be

$$m_x(C \cap C') = \dim_k \mathcal{O}_x/(f, g).$$

The intersection number $C \cdot C'$ is defined by

$$C \cdot C' = \sum_{x \in C \cap C'} m_x(C \cap C').$$

This may seem a strange definition of intersection number, so let me do a couple of examples: $y = 0$ intersect $y = x^2$; $y = x^2$ intersect $xy(x+y) = 0$ (admittedly the latter isn't irreducible). Various lemmas become plausible, including **Exercise, using what is coming.**

$$(f = 0) \cdot (gh = 0) = (f = 0) \cdot (g = 0) + (f = 0) \cdot (h = 0).$$

Then other calculations are easier, e.g. $(y^2 - x^3) \cdot (x^2 - y^3) = 4$.

Linear equivalence of divisors should be a homotopy, so it shouldn't be surprising that we have a map $\operatorname{Pic} S \times \operatorname{Pic} S \to \mathbb{Z}$.

Definition. For $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic} S$, define

$$\mathcal{L} \cdot \mathcal{L}' = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^*) - \chi(\mathcal{L}'^*) + \chi(\mathcal{L}^* \otimes \mathcal{L}'^*).$$

Theorem. If C and C' are two distinct irreducible curves on S then $\mathcal{O}_S(C) \cdot \mathcal{O}_S(C') = C \cdot C'$.

Proof. We'll show the exactness of

$$0 \to \mathcal{O}_S(-C - C') \stackrel{(s', -s)}{\to} \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \stackrel{(s, s')}{\to} \mathcal{O}_S \to \mathcal{O}_{C \cap C'} \to 0.$$

(Explain that $C \cap C'$ is taken scheme-theoretically.)

If we knew this, then taking Euler-characteristics would give us the desired result. We can prove this locally. Let's prove that:

$$0 \to \mathcal{O}_x \stackrel{(g, -f)}{\to} \mathcal{O}_x^2 \stackrel{(f, g)}{\to} \mathcal{O}_X \to \mathcal{O}_X/(f, g) \to 0.$$

Compositions are all zero. Injectivity on the left is clear, and the right term is by definition the cokernel. So the main thing to show is part of exactness at the second term. If there is an a and b such that $af + bg = 0$, then there is some c such that $a = cg$ and $b = -cf$. This follows from the fact that $\mathcal{O}_x$ is a unique factorization domain (fact: this is true of all local rings of smooth varieties), and f and g are relatively prime (as if they had a common factor, that would be a component of their intersection. $\square$

Proposition. If C is a non-singular irreducible curve on S, and $\mathcal{L} \in \operatorname{Pic} S$, then $\mathcal{O}_S(C) \cdot \mathcal{L} = \deg \mathcal{L}|_C$.

Interesting case: if $C \cdot C$ is thus interpreted as $\deg \mathcal{O}_S(C)|_C$, which is the degree of the normal bundle.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

Taking Euler characteristics gives $\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) = \chi(\mathcal{O}_C)$. Twist the exact sequence by $\mathcal{L}^*$, to get

$$0 \to \mathcal{L}^*(-C) \to \mathcal{L}^* \to \mathcal{L}^* \otimes \mathcal{O}_C \to 0,$$

and take Euler characteristics again to get $\chi(\mathcal{L}^*) - \chi(\mathcal{L}^*(-C)) = \chi(\mathcal{L}^*|_C)$. Thus

$$\begin{aligned} \mathcal{O}_S(C) \cdot \mathcal{L} &= \chi(\mathcal{O}_C) - \chi(\mathcal{L}^*|_C) \\ &= -\deg \mathcal{L}^*|_C \text{ by Riemann-Roch} \\ &= \deg \mathcal{L}|_C \end{aligned}$$

$\square$

Theorem. This is a symmetric bilinear form.

Symmetry is obvious. We'll prove bilinearity next time, using:

Fact (Serre). If D is any divisor on S and H is a hyperplane section of S (i.e. a section of a very ample line bundle). Then for $n >> 0$, $D + nH$ is also a very ample.

In particular, D can be written as $(D + nH) - (nH)$, the difference of two smooth curves on S. (Here we use Bertini's theorem.)

You should think of this as a moving lemma.