Recap of last time. Extended example of $\mathbb{P}^n$. $\mathcal{O}_{\mathbb{P}^n}(d) \leftrightarrow$ degree $d$ homogeneous things.

Maps to projective space corresponded to vector spaces of sections of an invertible sheaf $\mathcal{L}$ that are basepoint free (no common zero). Hyperplane sections correspond to $H^0(X, \mathcal{L})$.

For example, if the sections of $\mathcal{L}$ have no common zero, then we can map to some projective space by the vector space of all sections. Then we say that the invertible sheaf is $\mathcal{L}$ is basepoint free. (I didn’t give this definition last time.)

**Definition.** The corresponding map to projective space is called a linear system. (I’m not sure if I’ll use this terminology, but I want to play it safe.)

$$|\mathcal{L}| : X \to \mathbb{P}^n = \mathbb{P}H^0(X, \mathcal{L})^*.$$

**Definition.** An invertible sheaf $\mathcal{L}$ is very ample if the global sections of $\mathcal{L}$ gives a closed immersion into projective space.

**Fact.** equivalent to: “separates points and tangent vectors”.

**Definition.** An invertible sheaf is ample if some power of it is very ample.

**Note:** A very ample sheaf on a curve has positive degree. Hence an ample sheaf on a curve has positive degree. We’ll see later today that this is an “if and only if”.

*Date: Friday, October 11.*
Fact (Serre vanishing). Suppose \( M \) is any coherent sheaf e.g. an invertible sheaf, or more generally a locally free sheaf (essentially, a vector bundle), and \( \mathcal{L} \) is ample. Then for \( n >> 0 \), \( H^i(X, M \otimes \mathcal{L}^n) = 0 \) for \( i > 0 \).

1. Serre duality and Riemann-Roch; back to curves

Fact: Serre duality. If \( X \) is proper nonsingular and dimension \( n \), then for \( 0 \leq i \leq n \),
\[
H^i(X, \mathcal{L}) \otimes H^{n-i}(X, \mathcal{K} \otimes \mathcal{L}^\vee) \rightarrow H^n(X, \mathcal{K}) \sim \mathbb{C}
\]
is a perfect pairing.

(True for vector bundles. More general formulation for arbitrary coherent sheaves.) Thus \( h^i(X, \mathcal{L}) = h^{n-i}(X, \mathcal{K} \otimes \mathcal{L}^\vee) \) and \( \chi(C, \mathcal{L}) = (-1)^n \chi(C, \mathcal{K} \otimes \mathcal{L}^\vee) \).

In particular, we have Serre duality for curves. For any invertible sheaf \( \mathcal{L} \), the map
\[
H^0(C, \mathcal{L}) \otimes H^1(C, \mathcal{K} \otimes \mathcal{L}^\vee) \rightarrow H^1(C, \mathcal{K}) \sim \mathbb{C}
\]
is a perfect pairing.

Hence two possible definitions of genus are the same: \( h^0(C, \mathcal{K}) \) and \( h^1(C, \mathcal{O}_C) \).

Fact: Riemann-Roch Theorem.
\[
h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg(\mathcal{L}) - g + 1.
\]
(I.e. \( \chi(C, \mathcal{L}) = \deg - g + 1 \) — remember that the cohomology of a coherent sheaf vanishes above the dimension of the variety.)

Generalizations: Hirzebruch-Riemann-Roch, to Grothendieck-Riemann-Roch. Hirzebruch-Riemann-Roch, which we’ll be using, is a consequence of the Atiyah-Singer index theorem, which Rafe Mazzeo spoke about in the colloquium yesterday.

Proof of Riemann-Roch: (i) algebraic, (ii) complex-analytic, (iii) Atiyah-Singer.

Using Serre duality:
\[
h^0(C, \mathcal{L}) - h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee) = \deg(\mathcal{L}) - g + 1.
\]

Corollary. \( \deg K = 2g - 2 \). (Do it.)

So we have another definition of \( g \).

2. Applications of Riemann-Roch

Theorem. If \( \deg \mathcal{L} \geq 2g - 1 \), then \( h^1(C, \mathcal{L}) = 0 \). Hence \( h^0(C, \mathcal{L}) = \deg \mathcal{L} - g + 1 \).

Proof: Serre duality, and the fact that invertible sheaves of negative degree have no sections. Then Riemann-Roch.
**Theorem (numerical criterion for basepoint freeness).** If \( \deg L \geq 2g \), then \( L \) is basepoint free. (Remind them of definition.)

**Proof:** \( H^0(C, L(-p)) \) is the vector space of sections with a zero at \( p \). Want to show that \( H^0(C, L) - H^0(C, L(-p)) > 0 \). Do it using previous result: difference is 1!

**Example.** If \( C \) is genus 1 and \( L \) is degree 2, then we get a map to \( \mathbb{P}^1 \). This shows that every genus 1 curve is a double cover of \( \mathbb{P}^1 \). (Explain “hyperplane section” in this case.)

2.1. **Classification of genus 2 curves. Theorem.** Any genus 2 curve has a unique double cover of \( \mathbb{P}^1 \) branched over 6 points. (The 6 points are unique up to automorphisms of \( \mathbb{P}^1 \).) Any double cover of \( \mathbb{P}^1 \) branched over 6 points comes from a genus 2 curve. Hence genus 2 curves are classified by the space of 6 distinct points of \( \mathbb{P}^1 \), modulo automorphisms of \( \mathbb{P}^1 \). In particular, there is a dimension \( 6 - \dim \text{Aut} \mathbb{P}^1 = 3 \) moduli space.

**Proof.** First, every genus 2 curve \( C \) has a degree 2 map to \( \mathbb{P}^1 \) via \( K \) (basepoint freeness). The numerical criterion for basepoint freeness doesn’t apply, unfortunately. Let’s check that \( K \) has 2 sections: \( h^0(K) - h^0(O) = \deg(K) - 2 + 1 = 1 \). Next let’s check that \( K \) is basepoint free. We want \( K(-p) \) has only 1 section for all \( p \). If \( K(-p) \) had two sections, then we’d have a degree 1 map to \( \mathbb{P}^1 \) from \( C \) (explain), contradiction.

Next we’ll see that the only degree 2 maps to \( \mathbb{P}^1 \) arise from the canonical bundle. Suppose \( L \) is degree 2, and has 2 sections. Then by RR, \( K \otimes L^* \) has 1 section, and is degree 0. But the only degree 0 sheaf with a section is the trivial sheaf.

Finally, any double cover of \( \mathbb{P}^1 \) branched over 6 points is genus 2. This follows from the Riemann-Hurwitz formula.

Using naive geometry: \( 2 \chi(\mathbb{P}^1) - 6 = \chi(C) \), i.e. \( -2 = 2 - 2g \), so \( g = 2 \).

**Fact.** Riemann-Hurwitz formula: \( K_C = \pi^*K_{\mathbb{P}^1} + \text{ramification divisor} \). Taking degrees: \( 2g - 2 = 2(-2) + 6 \).

2.2. **A numerical criterion for very ampleness. Theorem.** If \( \deg L \geq 2g + 1 \), then \( L \) is very ample. (Remind them of definition.)

**Proof.** I’ll show you that it separates points. Suppose \( p \neq q \). \( H^0(C, L) - H^0(C, L(-p - q)) = 2 \). Hence there is a section vanishing at \( p \) and not at \( q \), and vice versa.

Same idea works for separating tangent vectors. \( h^0(C, L) - h^0(C, L(-2p)) = 2 \). Hence there is a section vanishing at \( p \) but not to order 2.

**Example.** \( C \) genus 1, \( L \) has degree 3. Then \( C \hookrightarrow \mathbb{P}^2 \), degree 3. Hence every elliptic curve can be described by a plane cubic. (We’ll soon see that any plane cubic is a genus 1 curve.)

**Corollary.** \( L \) on \( C \) is ample iff it has positive degree.
There is one more natural place to find line bundles: the normal bundle to a submanifold of dimension 1 less. I’ll now discuss normal bundles algebraically, and as an application prove the adjunction formula.

Suppose \( D \) is a nonsingular divisor (codimension 1) on nonsingular \( X \). Then we can understand the canonical sheaf of \( D \) in terms of the canonical sheaf of \( X \).

**The adjunction formula.** \( \mathcal{K}_D = \mathcal{K}_X(D)|_D \).

( Remind them what \( \mathcal{K}_X(D) \) means. )

This actually holds in much more generality, e.g. \( D \) can be arbitrarily singular, and \( X \) need not be smooth.

Here is an informal description of why this is true.

- Motivation: tangent space in differential geometry to a point \( p \) in a manifold \( W \) is the space of curves, modulo some equivalence relation.

- The cotangent space is the dual of this space, and can be interpreted as the space of functions vanishing at \( p \) modulo an equivalence relation: the functions vanishing to order 2 at \( p \). In the space \( \mathcal{O} \) of functions near \( p \), the first is the ideal \( I_p \), the second is the ideal \( I_p^2 \). Thus the cotangent space is \( I_p/I_p^2 \).

- Aside: in this case, \( I_p \) is a maximal ideal \( m_p \), as \( \mathcal{O}/I_p \) is a field. So the cotangent space is \( m_p/m_p^2 \). That’s why the Zariski tangent space is defined as \( m_p/m_p^2 \). This is a purely algebraic definition. It works for any algebraic \( X \), not even defined over a field, not even non-singular.

- Next consider \( Y \) to be a nonsingular subvariety of \( X \), of codimension \( c \). Then there is a conormal bundle of \( Y \) in \( X \) (the dual of the normal bundle. It has rank \( d \). The conormal sheaf is the sheaf of sections of \( Y \). It is locally free of rank \( d \). **Definition (motivated by previous discussion).** In sheaf language: the conormal bundle is \( \mathcal{I}/\mathcal{I}^2 \). This is a priori a sheaf on all of \( X \), but in fact it lives on \( Y \) (“is supported on \( Y \”).

- Now suppose further that \( Y \) is a divisor, so I’ll now call it \( D \). Then \( \mathcal{I} = \mathcal{O}_X(D) \). We’re modding out this sheaf by functions vanishing on \( D \); this is the same as restricting to \( D \). Hence \( \mathcal{I}/\mathcal{I}^2 \sim \mathcal{O}(D)|_D \).

**Proof of the adjunction formula.** We have an exact sequence

\[
0 \to T_D \to T_X|_D \to N_D \to 0.
\]

Dualize to get:

\[
0 \to N_D^* \to \Omega_X|_D \to \Omega_D \to 0.
\]

Take top wedge powers to get \( \mathcal{K}_X|_D \sim \mathcal{K}_D \otimes N_D^* \) from which

\[
\mathcal{K}_D \sim \mathcal{K}_X|_D \otimes N_D = \mathcal{K}_X|_D \otimes \mathcal{O}_X(D)|_D = \mathcal{K}_X(D)|_D.
\]
3.1. **Applications of the adjunction formula.** 1) Cubics in $\mathbb{P}^2$ have trivial canonical bundle. Hence all genus 1 curves have canonical sheaf that is not degree 0, but also trivial.

2) Quartics in $\mathbb{P}^3$ also have trivial canonical bundle. K3 surfaces.

3) What’s the genus of a smooth degree $d$ curve in $\mathbb{P}^3$? Answer: $(d - 1)(d - 2)/2$.

4) Smooth quartics in $\mathbb{P}^2$ are embedded by their canonical sheaf. More on this in a second.

5) Smooth complete intersection of surfaces of degree 2 and 3 in $\mathbb{P}^3$: also embedded by their canonical sheaf.

3.2. **Classification of curves of genus 3.** I’ll discuss at greater length next day.

**Theorem.** Every smooth curve of genus 3 is of (precisely) one of the two following forms.

(i) A smooth quartic curve in $\mathbb{P}^2$ (in a unique way, up to automorphisms of $\mathbb{P}^2$).

(ii) A double cover of $\mathbb{P}^1$ branched over 8 points (in a unique way, up to automorphisms of $\mathbb{P}^2$).