COMPLEX ALGEBRAIC SURFACES CLASS 4

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CONTENTS

Recap of last time. Extended example of $\mathbb{P}^n$. $\mathcal{O}_{\mathbb{P}^n}(d) \leftrightarrow$ degree d homogeneous things.

Maps to projective space corresponded to vector spaces of sections of an invertible sheaf $\mathcal{L}$ that are *basepoint free* (no common zero). Hyperplane sections correspond to $H^0(X, \mathcal{L})$.

For example, if the sections of $\mathcal{L}$ have no common zero, then we can map to some projective space by the vector space of *all* sections. Then we say that *the invertible sheaf is $\mathcal{L}$ is basepoint free.* (I didn't give this definition last time.)

Definition. The corresponding map to projective space is called a *linear system*. (I'm not sure if I'll use this terminology, but I want to play it safe.)

$$|\mathcal{L}| : X \to \mathbb{P}^n = \mathbb{P}H^0(X, \mathcal{L})^*.$$

Definition. An invertible sheaf $\mathcal{L}$ is *very ample* if the global sections of $\mathcal{L}$ gives a closed immersion into projective space.

Fact. equivalent to: "separates points and tangent vectors".

Definition. An invertible sheaf is *ample* if some power of it is very ample.

Note: A very ample sheaf on a curve has positive degree. Hence an ample sheaf on a curve has positive degree. We'll see later today that this is an "if and only if".

Date: Friday, October 11.

Fact (Serre vanishing). Suppose $\mathcal{M}$ is any coherent sheaf e.g. an invertible sheaf, or more generally a locally free sheaf (essentially, a vector bundle), and $\mathcal{L}$ is *ample*. Then for $n >> 0$, $H^i(X, \mathcal{M} \otimes \mathcal{L}^n) = 0$ for $i > 0$.

1. Serre duality and Riemann-Roch; back to curves

Fact: Serre duality. If X is proper nonsingular and dimension n, then for $0 \leq i \leq n$,

$$H^i(X, \mathcal{L}) \otimes H^{n-i}(X, \mathcal{K} \otimes \mathcal{L}^\vee) \to H^n(X, \mathcal{K}) \sim \mathbb{C}$$

is a perfect pairing.

(True for vector bundles. More general formulation for arbitrary coherent sheaves.) Thus $h^i(X, \mathcal{L}) = h^{n-i}(X, K \otimes \mathcal{L}^*)$ and $\chi(C, \mathcal{L}) = (-1)^n \chi(C, \mathcal{K} \otimes \mathcal{L}^\vee)$.

In particular, we have **Serre duality for curves.** For any invertible sheaf $\mathcal{L}$, the map

$$H^0(C, \mathcal{L}) \otimes H^1(C, \mathcal{K} \otimes \mathcal{L}^\vee) \to H^1(C, \mathcal{K}) \sim \mathbb{C}$$

is a perfect pairing.

Hence two possible definitions of genus are the same: $h^0(C, \mathcal{K})$ and $h^1(C, \mathcal{O}_C)$.

Fact: Riemann-Roch Theorem.

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg(\mathcal{L}) - g + 1.$$

(I.e. $\chi(C, \mathcal{L}) = \deg - g + 1$ — rememeber that the cohomology of a coherent sheaf vanishes above the dimension of the variety.)

Generalizations: Hirzebruch-Riemann-Roch, to Grothendieck-Riemann-Roch. Hirzebruch-Riemann-Roch, which we'll be using, is a consequence of the Atiyah-Singer index theorem, which Rafe Mazzeo spoke about in the colloquium yesterday.

Proof of Riemann-Roch: (i) algebraic, (ii) complex-analytic, (iii) Atiyah-Singer.

Using Serre duality:

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee) = \deg(\mathcal{L}) - g + 1.$$

Corollary. $\deg K = 2g - 2$. (Do it.)

So we have another definition of g.

2. Applications of Riemann-Roch

Theorem. If $\deg \mathcal{L} \geq 2g - 1$, then $h^1(C, \mathcal{L}) = 0$. Hence $h^0(C, \mathcal{L}) = \deg \mathcal{L} - g + 1$.

Proof: Serre duality, and the fact that invertible sheaves of negative degree have no sections. Then Riemann-Roch.

Theorem (numerical criterion for basepoint freeness). If $\deg \mathcal{L} \geq 2g$, then $\mathcal{L}$ is basepoint free. (Remind them of definition.)

Proof: $H^0(C, \mathcal{L}(-p))$ is the vector space of sections with a zero at p. Want to show that $H^0(C, \mathcal{L}) - H^0(C, \mathcal{L}(-p)) > 0$. Do it using previous result: difference is 1!

Example. If C is genus 1 and $\mathcal{L}$ is degree 2, then we get a map to $\mathbb{P}^1$. This shows that every genus 1 curve is a double cover of $\mathbb{P}^1$. (Explain "hyperplane section" in this case.)

2.1. **Classification of genus 2 curves. Theorem.** Any genus 2 curve has a unique double cover of $\mathbb{P}^1$ branched over 6 points. (The 6 points are unique up to automorphisms of $\mathbb{P}^1$.) Any double cover of $\mathbb{P}^1$ branched over 6 points comes from a genus 2 curve. Hence genus 2 curves are classified by the space of 6 distinct points of $\mathbb{P}^1$, modulo automorphisms of $\mathbb{P}^1$. In particular, there is a dimension $6 - \dim \operatorname{Aut} \mathbb{P}^1 = 3$ moduli space.

Proof. First, every genus 2 curve C has a degree 2 map to $\mathbb{P}^1$ via $\mathcal{K}$ (basepoint freeness). The numerical criterion for basepoint freeness doesn't apply, unfortunately. Let's check that $\mathcal{K}$ has 2 sections: $h^0(\mathcal{K}) - h^0(\mathcal{O}) = \deg(\mathcal{K}) - 2 + 1 = 1$. Next, let's check that $\mathcal{K}$ is basepoint free. We want $\mathcal{K}(-p)$ has only 1 section for all p. If $\mathcal{K}(-p)$ had two sections, then we'd have a degree 1 map to $\mathbb{P}^1$ from C (explain), contradiction.

Next we'll see that the only degree 2 maps to $\mathbb{P}^1$ arise from the canonical bundle. Suppose $\mathcal{L}$ is degree 2, and has 2 sections. Then by RR, $\mathcal{K} \otimes \mathcal{L}^*$ has 1 section, and is degree 0. But the only degree 0 sheaf with a section is the trivial sheaf.

Finally, any double cover of $\mathbb{P}^1$ branched over 6 points is genus 2. This follows from the *Riemann-Hurwitz formula*.

Using naive geometry: $2\chi(\mathbb{P}^1) - 6 = \chi(C)$, i.e. $-2 = 2 - 2g$, so $g = 2.$.

Fact. Riemann-Hurwitz formula: $\mathcal{K}_C = \pi^* \mathcal{K}_{\mathbb{P}^1} +$ ramification divisor. Taking degrees: $2g - 2 = 2(-2) + 6.$ $\square$

2.2. **A numerical criterion for very ampleness. Theorem.** If $\deg \mathcal{L} \geq 2g + 1$, then $\mathcal{L}$ is very ample. (Remind them of definition.)

Proof. I'll show you that it separates points. Suppose $p \neq q$. $H^0(C, \mathcal{L}) - H^0(C, \mathcal{L}(-p - q)) = 2$. Hence there is a section vanishing at p and not at q, and vice versa.

Same idea works for separating tangent vectors. $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 2$. Hence there is a section vanishing at p but not to order 2.

Example. C genus 1, $\mathcal{L}$ has degree 3. Then $C \hookrightarrow \mathbb{P}^2$, degree 3. Hence every elliptic curve can be described by a plane cubic. (We'll soon see that any plane cubic is a genus 1 curve.)

Corollary. $\mathcal{L}$ on C is ample iff it has positive degree.

3. Normal bundles; the adjunction formula

There is one more natural place to find line bundles: the normal bundle to a submanifold of dimension 1 less. I'll now discuss normal bundles algebraically, and as an application prove the *adjunction formula*.

Suppose D is a nonsingular divisor (codimension 1) on nonsingular X. Then we can understand the canonical sheaf of D in terms of the canonical sheaf of X.

The adjunction formula. $\mathcal{K}_D = \mathcal{K}_X(D)|_D$.

(Remind them what $\mathcal{K}_X(D)$ means.)

This actually holds in much more generality, e.g. D can be arbitrarily singular, and X need not be smooth.

Here is an informal description of why this is true.

• Motivation: tangent space in differential geometry to a point p in a manifold W is the space of curves, modulo some equivalence relation.

• The cotangent space is the dual of this space, and can be interpreted as the space of functions vanishing at p modulo an equivalence relation: the functions vanishing to order 2 at p. In the space $\mathcal{O}$ of functions near p, the first is the ideal I_p, the second is the ideal I_p^2. Thus the cotangent space is I_p/I_p^2.

• Aside: in this case, I_p is a maximal ideal $\mathfrak{m}_p$, as $\mathcal{O}/I_p$ is a field. So the cotangent space is $\mathfrak{m}_p/\mathfrak{m}_p^2$. That's why the Zariski tangent spaced is defined as $\mathfrak{m}_p/\mathfrak{m}_p^2$. This is a purely algebraic definition. It works for any algebraic X, not even defined over a field, not even non-singular.

• Next consider Y to be a nonsingular subvariety of X, of codimension c. Then there is a conormal bundle of Y in X (the dual of the normal bundle. It has rank d. The conormal sheaf is the sheaf of sections of Y. It is locally free of rank d. *Definition (motivated by previous discussion).* In sheaf language: the conormal bundle is $\mathcal{I}/\mathcal{I}^2$. This is a priori a sheaf on all of X, but in fact it lives on Y ("is supported on Y").

• Now suppose further that Y is a divisor, so I'll now call it D. Then $\mathcal{I} = \mathcal{O}_X(D)$. We're modding out this sheaf by functions vanishing on D; this is the same as restricting to D. Hence $\mathcal{I}/\mathcal{I}^2 \sim \mathcal{O}(D)|_D$.

Proof of the adjunction formula. We have an exact sequence

$$0 \to T_D \to T_X|_D \to N_D \to 0.$$

Dualize to get:

$$0 \to N_D^* \to \Omega_X|_D \to \Omega_D \to 0.$$

Take top wedge powers to get $\mathcal{K}_X|_D \sim \mathcal{K}_D \otimes N_D^*$ from which

$$\mathcal{K}_D \sim \mathcal{K}_X|_D \otimes N_D = \mathcal{K}_X|_D \otimes \mathcal{O}_X(D)|_D = \mathcal{K}_X(D)|_D.$$

□

3.1. **Applications of the adjunction formula.**

1) Cubics in $\mathbb{P}^2$ have trivial canonical bundle. Hence all genus 1 curves have canonical sheaf that is not degree 0, but also trivial.

2) Quartics in $\mathbb{P}^3$ also have trivial canonical bundle. K3 surfaces.

3) What's the genus of a smooth degree d curve in $\mathbb{P}^3$? Answer: $(d-1)(d-2)/2$.

4) Smooth quartics in $\mathbb{P}^2$ are embedded by their canonical sheaf. More on this in a second.

5) Smooth complete interesction of surfaces of degree 2 and 3 in $\mathbb{P}^3$: also embedded by their canonical sheaf.

3.2. **Classification of curves of genus 3.**

I'll discuss at greater length next day.

Theorem. Every smooth curve of genus 3 is of (precisely) one of the two following forms.

(i) A smooth quartic curve in $\mathbb{P}^2$ (in a unique way, up to automorphisms of $\mathbb{P}^2$).

(ii) A double cover of $\mathbb{P}^1$ branched over 8 points (in a unique way, up to automorphisms of $\mathbb{P}^2$).