COMPLEX ALGEBRAIC SURFACES CLASS 2

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CONTENTS

If you didn't get an e-mail from me, you're not on the e-mail list, so let me know. (If you later want to be off the e-mail list, also let me know.) Change in time: Starting today, we'll meet Wednesdays and Fridays 2:10–3:25. We might meet on a couple of Mondays as well for 50 minutes to catch up if need be; if that happens, I'll warn you in advance. There will be no class this coming Friday.

A few comments for those who are more comfortable in the analytic topology for complex varieties: One on hand, feel free to interpret everything I say in that topology. In general, you won't go wrong, and if you're in doubt, just ask. On the other hand, you may as well think in the Zariski topology; essentially everything you know applies in the algebraic category. This is a consequence of Serre's GAGA theorem, which loosely says that in a wide variety of situations, the result of any calculation in the analytic topology for complex projective varieties is the same as the same calculation in the algebraic category. For example, cohomology groups of coherent sheaves; also they can be computed using Cech cohomology in both categories. Also: invertible sheaves (line bundles). Again, if you're in doubt, or you're curious, just ask.

Where we were last day. Our short-term goal is to classify non-singular curves, and in the process to pick up useful techniques that we'll use repeatedly later. In particular, We were in the process of discussing invertible sheaves (aka line bundles) and the Picard group. Let's continue our discussion of invertible sheaves, and later take it back to curves.

Date: Wednesday, October 2.

1. Invertible sheaves = Line bundles = $H^1(X, \mathcal{O}_X^*)$, and Pic(X)

For now, X is nonsingular of any dimension, and we're working over any field. Much of this discussion generalizes further, to singular schemes; the main difference is that the issue of divisors becomes more subtle, and I don't want to get into that.

The invertible sheaves form a group $\text{Pic}(X)$, called the Picard group, under tensor product.

André pointed out an alternative definition of $\text{Pic}(X)$, $H^1(X, \mathcal{O}_X^*)$. Here's the link.

An invertible sheaf $\mathcal{L}$ is locally trivial. That means that there's an open cover $(U_i)_{1 \leq i \leq n}$ of X over which the sheaf is trivial. Let $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$. There is gluing data $g_{ij} : \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ that corresponds to multiplying by an invertible function $g_{ij} \in H^0(U_{ij})$, such that $g_{ij} g_{jk} = g_{ik}$ on U_{ijk}.

You can interpret the sections of the sheaf over an open set U in terms of this data. A section s corresponds to the data of sections over each open set $s_i \in H^0(U_i, \mathcal{O})$ such that $s_j = g_{ij} s_i$ on U_{ij}.

This data is precisely a Cech cocycle for $\mathcal{O}_X^*$.

Exercise. Show that each invertible sheaf defines a well-defined element of $H^1(X, \mathcal{O}_X^*)$, and vice versa.

Remark. You can pull back invertible sheaves.

1.1. The group of line bundles with non-zero rational section is the group of divisors.

As I mentioned yesterday, there if s is a rational (meromorphic) section of $\mathcal{L}$ and t is a rational section of $\mathcal{M}$, then st is a rational section of $\mathcal{L} \otimes \mathcal{M}$ and s/t is a rational section of $\mathcal{L} \otimes \mathcal{M}^\vee$.

Hence there's a group ($\mathcal{L} \in \text{Pic}X$, s rat'l section).

To each *non-zero* rational section s, you get a *divisor* $div(s) := \oplus_{Y \subset X} n_Y Y$. Finitely many non-zero; Y are the codimension 1 (irreducible) subvarieties. n_Y is the order of 0 or (if negative) pole. For example, consider the structure sheaf on $\mathbb{A}^2$, and the section $xy(y - x^2)/(y - x^3)$. The corresponding divisor is $(x) + (y) + (y - x^2) - (y - x^3)$.

If you take a multiple of this section by a function that has no zeros or poles (e.g. a non-zero constant), you get the same divisor. Hence we get a map:

$$(\mathcal{L} \in \text{Pic}X, s \text{ rat'l section})/\text{functions with no zeros or poles} \to \text{divisors}. \tag{1}$$

Another *fact:* for a proper (e.g. projective) variety, the only functions are constant functions; in complex geometry, this is the maximum principle. So (1) can be rewritten

$$(\mathcal{L} \in \mathrm{Pic}(X), s \text{ rat'l section})/k^* \leftrightarrow \text{divisors.} \tag{2}$$

In fact, this is reversible (so $\rightarrow$ can be replaced by $\leftrightarrow$). Warning: This is subtle! Examples will come soon.

To the divisor $D := \oplus_Y n_Y Y$, we associate a sheaf $\mathcal{O}(D)$. To describe it, I need to tell you what the sections over any open set U, denoted $H^0(U, \mathcal{O}(D)) = \Gamma(U, \mathcal{O}(D))$, are. They are the functions f whose divisor $div(f) + D \geq 0$ (explain). I also need to give you a rational section. This is the section 1.

It is not immediately clear first of all that $\mathcal{O}(D)$ is an invertible sheaf, that locally it is trivial. Take a point $x \in X$; then (fact) in a small enough neighbourhood of X, D is a *principal divisor*, i.e. the divisor of a rational function. (In other words, there's a neighbourhood $U \subset X$ containing x and a rational function f on U such that $D|_U = div(f)$.) Then the trivialization is given by

$$\begin{array}{rcl} \text{sheaf } \mathcal{O}_U(D) & \leftrightarrow & \text{sheaf of functions on } U \\ \text{section } s \in H^0(V, \mathcal{O}_U(D)) & \leftrightarrow & sf \in H^0(V, \mathcal{O}_U). \end{array}$$

In particular, the divisor of the section corresponding to s isn't $div(s)$; it is the divisor $div(s/f)$.

Again, an example will be coming soon ($\mathbb{P}^1$), which may be somewhat helpful.

If $\mathcal{L}$ is an invertible sheaf and D is a divisor, we can define $\mathcal{L}(D)$ similarly; $\mathcal{L}(D) \equiv \mathcal{L} \otimes \mathcal{O}(D)$.

1.2. **The Picard group is the group of divisors modulo linear equivalence.** Next, suppose we take two non-zero rational sections s_1 and s_2 of $\mathcal{L}$. They will give different divisors D_1 and D_2. How are they related? Well, s_1/s_2 is a rational section of the structure sheaf $\mathcal{O}_X$, i.e. a rational function. Hence we get

$$\mathrm{Pic}(X) \leftrightarrow \text{divisors/divisors of rational functions.}$$

Two divisors whose difference is a divisor of a rational function are said to be *linearly equivalent*.

This lets us calculate Pic in some cases. For example: **Exercise:** Show that $\mathbb{A}^1 = \operatorname{Spec} k[t]$ has trivial Pic. Here's how: you want to show that any divisor is a divisor of a rational function. (Do an example.)

1.3. **An extended example: $\mathbb{P}^1$.** Co-ordinates on $\mathbb{P}^1$: $[x_0; x_1]$. (Informal language: $[x; 1]$ is called the point x; $[1; 0]$ is called the point at ∞, or just plain ∞.) $\mathbb{P}^1$ is the union of two open sets. $U_0 : x_1 \neq 0$, with coordinate x: $[x; 1]$. $U_1 : x_0 \neq 0$, with coordinate y, $[1; y]$. x and y are rational functions. On intersection $U_{01} := U_0 \cap U_1$ (or as rational functions), $x = 1/y$.

1) Interpret $\mathcal{O}([0; 1])$ in $\mathbb{P}^1$.

First, let's find the space of global sections. Restrict to U_1: regular functions in y: polynomials in y. Restrict to U_0: rational functions in x that are regular except at 0 you're allowed a pole. $a/x + f(x)$. Which ones glue? Answer: functions of the form $a/x + b$ on U_0 turn into $ay + b$ on U_1. $h^0(\mathbb{P}^1, \mathcal{O}([0; 1])) = 2$. (**Exercise:** Show that $H^0(\mathbb{P}^1, n\mathcal{O}([0; 1])) = n+1$ for $n \geq -1$.

Next question: the section 1. Where does it have poles and zeros? Answer: it has a 0 at $[0; 1]$. How about $1/x$? Answer: it has a 0 at $[1; 0]$.

Next question: rational section described by $x(x-1)^2(x-3)$. Where are the poles and zeros? (There are five poles and four zeros.)

Note: the number of poles minus the number of zeros is always 1. **Exercise:** Prove this for all rational sections. (We'll answer this soon.)

2) Interpret $\mathcal{O}(-[0; 1])$ on $\mathbb{P}^1$. How many global sections? Only the zero-section (do this). Find a rational section, and calculate number of poles and zeros.

3) **Exercise:** Show that any two points on $\mathbb{P}^1$ are linearly equivalent. (Do a few specific cases, e.g. 2 and 3.)

Hence show that Pic$\mathbb{P}^1$ is generated as a group by the divisor $\mathcal{O}_{\mathbb{P}^1)}(1) := \mathcal{O}_{\mathbb{P}^1}(pt)$.

Then you've almost shown that Pic$\mathbb{P}^1$ $= \mathbb{Z}$; all that's left is to show that $\mathcal{O}_{\mathbb{P}^1}(d)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^1}$ for any $d \neq 0$.

Exercise: Show that the space of global sections of $\mathcal{O}_{\mathbb{P}^1}(d)$ is $(d+1)$-dimensional for $d \geq 0$. (Hint: this was an earlier exercise: consider $\mathcal{O}_{\mathbb{P}^1}(d[0; 1])$.)

Another way to show that $\mathcal{O}(d)$ is not isomorphic to $\mathcal{O}$:

Definition. Assume (for this definition) that X is a (proper smooth) curve. Degree of an invertible sheaf on a smooth proper curve: take a non-zero rational section, and count zeros minus poles. (To show that this is well-defined, we need only check that the degree of a rational function is always 0. Take this as a given *fact* if you want. In complex geometry, you can show this using Stokes' theorem.) Important if trivial consequences: a sheaf of negative degree has no sections. A sheaf of degree 0 has no sections unless it is the structure sheaf (the sheaf of functions).

Definition. The canonical sheaf, denoted $\mathcal{K}_X$: top wedge product of cotangent sheaf.

So far the only examples of line bundles we've seen are $\mathcal{O}_X$ (i.e. functions), and $\mathcal{O}_{\mathbb{P}^1}(d)$. Here is another important one.

Example. $\mathcal{K}_{\mathbb{P}^1} = \mathcal{O}(-2)$: dx is a rational section; $dy = -(dx)/y^2$, pole of order 2 at ∞. Using complex geometry, $\mathcal{K}_E$ (E genus 1 curve) is trivial: there is a differential dz with no zeros or poles. (More generally, this is true of abelian varieties.)