COMPLEX ALGEBRAIC SURFACES CLASS 1

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CONTENTS

1. INTRODUCTION

In this course, I intend to cover as much of Enriques' classification of algebraic surfaces as possible.

This might be of interest to: algebraic geometers, low-dimensional topologists and symplectic geometers, certain arithmetic geometers, possibly physicists.

Classification of surfaces useful for various reasons. Model for classifications in general. Flavor carries over. Also useful: you can have a surface. En route useful things to learn: geometry of surfaces.

As far as possible, I want to stick to algebro-geometric techniques. A subtext is to give you some familiarity with many ideas in algebraic geometry. On the other hand, I won't assume that you've seen much algebraic geometry, although a first course would help.

(Go through the handout.) Prerequites. Books: Beauville, Hartshorne, Reid. I intend to type up notes from as many of the classes as possible, and post them on the course webpage: `http://www.math.stanford.edu/~vakil/245/`. They won't be especially refined.

The timing of the class will change to Wednesdays and Fridays 2:05–3:20. We may still have another couple of Mondays at a later date.

Date: Monday, September 30.

2. Classification of non-singular algebraic curves

Today, I want to discuss the classification of smooth curves, for two reasons. First, it anticipates some of the ideas that will come up in surfaces, and we'll see some of the characteristics of the classification, including weaknesses. Second, it will give me a chance to introduce some background. Classification of surfaces requires some knowledge of curves, as well as certain results based on curves.

As far as possible, we'll work over an arbitrary algebraically closed field $k = \overline{k}$, but for now let me discuss Riemann surfaces. All curves during this course will be proper, smooth, connected, unless otherwise specified.

Compact connected oriented 2-dimensional manifolds, with a complex structure. All Riemann surfaces are projective. This is *not* true in higher dimensions in general.

Other special facts about curves.

(1) Suppose C is a smooth curve, and $U \subset C$ is an open set. Then any map $U \to \mathbb{P}^n$ extends to $C \to \mathbb{P}^n$. This is true because of the *valuative criterion of properness*, although it can be explained more simply. Intuitively, one-parameter limits exist. This is false in higher dimension. For example, consider $U \subset \mathbb{A}^2$, where $U = \mathbb{A}^2 - (0,0)$. Consider $U \to \mathbb{P}^1$ given by $(x, y) \mapsto [x; y]$ (draw picture). Then there is no way to extend this map. **Exercise:** prove this. (Comment about exercises.)

(2) Hence two smooth projective curves are birational if and only if they are biregular. This is not true for surfaces; we'll see why.

(3) "Hence" there is a bijection between curves (up to birational or biregular isomorphism) and their function fields (fields of rational functions).

So curves, for these and other reasons, are nicer than surfaces.

3. Classification

Topologically easy: genus. In fact, the curves of given genus g form a nice family, in the right sense: a connect space of dimension $3g - 3$. If you define things correctly, it is connected and quasiprojective, i.e. an open subset of a projective variety.

Hence genus is the only discrete invariant.

Surfaces: topologically hard. Moduli of surfaces is also much more difficult; the existence of good moduli spaces of surfaces of general type was shown only in the past decade. The Enriques classification is more elementary. Also: analog in surfaces will be 0, 1, and > 1; there is a big change in behavior.

4. How to classify algebraic curves algebro-geometrically?

(This amounts to tying one hand behind our back.)

There is a complex line bundle corresonding to differentials, denoted $\mathcal{K}$ (the canonical bundle). $H^0(C, \mathcal{K}) = g$. Works in any dimension: top wedge of cotangent bundle.

In order to discuss this, I'll first talk about line bundles in the language of divisors and invertible sheaves, something absolutely central to our later discussions.

5. Invertible sheaves (= line bundles) and the Picard group

This is fundamental, and quite tricky if you haven't seen it before. After you've become comfortable with it, it will be quite natural. This language is essential to algebraic geometry (and this course in particular).

Let X be a *smooth variety*.

Language of invertible sheaves: sheaf locally isomorphic to structure sheaf (sheaf of functions, denoted by $\mathcal{O}$ or $\mathcal{O}_X$).

Line bundles correspond to invertible sheaves. Invertible sheaf := sheaf of sections of a line bundle. Line bundle := total space of an invertible sheaf.

If you prefer, you can think of transition functions.

Invertible sheaves form a group under tensor product (multiply transition functions). This group is called the *Picard group* $\operatorname{Pic} X$. The identity in the group is the structure sheaf $\mathcal{O}_X$. If s is a section of $\mathcal{L}$ and t is a section of $\mathcal{M}$, then st is a section of $\mathcal{L} \otimes \mathcal{M}$ and s/t is a section of $\mathcal{L} \otimes \mathcal{M}^\vee$.

To each non-zero rational section s, you get a *divisor* $div(s) := \oplus_{Y \subset X} n_Y Y$. Finitely many non-zero; Y are the codimension 1 (irreducible) subvarieties. n_Y is the order of 0 or (if negative) pole. For example, consider the structure sheaf on $\mathbb{A}^2$, and the section $xy(y - x^2)/(y - x^3)$. The corresponding divisor is $(x) + (y) + (y - x^2) - (y - x^3)$.

6. Next day:

André pointed out that another definition of $\operatorname{Pic}(X)$ is $H^1(X, \mathcal{O}_X^*)$. We'll see why this is the case.

Next, we'll relate $\operatorname{Pic}(X)$ and divisors on X. More precisely:

$$(\mathcal{L} \in \operatorname{Pic} X, s \text{ rat'l section})/\text{functions with no zeros or poles} \to \text{divisors}.$$

For proper (e.g. projective) X,

$$(\mathcal{L} \in \mathrm{Pic}(X), s \text{ rat'l section})/k^* \leftrightarrow \text{divisors}.$$

$\mathrm{Pic}(X)$ in terms of divisors:

$$\mathrm{Pic}(X) \leftrightarrow \text{divisors/divisors of rational functions}.$$

Then we'll find the Picard groups of $\mathbb{A}^1$ and $\mathbb{P}^1$, using the fact that invertible sheaves on projective curves have *degrees*.

Then we'll relate invertible sheaves to maps to projective space: base point freeness, ampleness, very ampleness. Serre vanishing, Serre duality. Then we'll go back to curves...