COMPLEX ALGEBRAIC SURFACES CLASS 19

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CONTENTS

Today, I'm going to sketch the Enriques classification of surfaces. I'm not going to prove any theorems.

1. Kodaira dimension

Definition. Let V be a smooth projective variety, ϕ_{nK} the rational map from V to the projective space associated with the linear system $|nK|$. The *Kodaira dimension* of V, written $\kappa(V)$, is the maximum dimension of the images $\phi_{nK}(V)$ for $n \geq 1$. If $|nK| = \emptyset$, then $\phi_{nK}(V) = \emptyset$, and we say $\dim \emptyset = -\infty$.

Hence $\kappa(V) \in \{-\infty, 0, 1, 2, \ldots, \dim V\}$.

Exercise. If V is a curve, then $V = -\infty$ if $g = 0$, $V = 0$ if $g = 1$, and $V = 1$ if $g \geq 2$.

Notice that this is a good trichotomy; there is a big change in behavior between these three classes of curves.

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In the case we're interested in, $\dim V = 2$, so there are four possible Kodaira dimensions.

The classification theorem of Enriques classifies all possibilities for which the $\kappa = -\infty$, 0, and 1. For 2, there are infinitely many families, and we can still say something interesting.

Exercise. Using the adjunction formula, show that the complete intersection of $n-2$ hypersurfaces in $\mathbb{P}^n$, of degrees $d_1, \ldots, d_{n-2}$, has Kodaira dimension $-\infty$ if $\vec{d} = (2)$, (3), or $(2,2)$; 0 if $\vec{d} = (4)$, $(2,3)$, or $(2,2,2)$ (examples of K3 surfaces), and 2 otherwise.

Exercise. If $S = C \times D$ where C and D are smooth curves, then

- If C or D is rational, then S is ruled, and $\kappa = -\infty$.
- If C and D are elliptic, $\kappa = 0$. (Abelian surface)
- If C is elliptic and $g(D) \geq 2$, then $\kappa(S) = 1$.
- If C and D are of genus at least 2, then $\kappa(S) = 2$.

Remark: two of the above are examples of *elliptic fibrations*. An elliptic fibration on X is a holomorphic map $f : X \to C$ where C is a curve, such that the general fiber of f is a smooth elliptic curve. An *elliptic surface* is a surface with a given elliptic fibration. You should think of this as a variation of "ruled surface".

Before I begin stating the classification, here is a fun fact, saying that the plurigenera "stabilize" by the *twelfth plurigenus*.

- $\kappa = -\infty$ iff $P_{12} = 0$.
- $\kappa = 0$ iff $P_{12} = 1$.
- $\kappa = 1$ iff $P_{12} \geq 2$ and $K^2 = 0$.
- $\kappa = 2$ iff $P_{12} \geq 2$ and $K^2 > 0$.

2. Kodaira dimension $-\infty$

Key theorem:

Enriques' Theorem. Let S be a surface with $P_4 = P_6 = 0$ (or $P_{12} = 0$). Then S is ruled.

Idea behind proof. If $q = 0$, we're done: $P_2 = 0$, so S is rational by Castelnuovo's criterion for rationality.

If $q \geq 1$, there is more work to be done, but it isn't harder than what we've done so far. How to start: The image of S in the Albanese is a curve, smooth of genus q. That will be the base of our ruled surface. $\square$

We already computed that if S is ruled then $P_n = 0$ for all n.

Thus: $\kappa(S) = -\infty$ iff $P_n = 0$ for all $n \geq 1$, iff (by Enriques' theorem) S is ruled.

3. Kodaira dimension 0

Theorem and Definition. Let S be a minimal surface with $\kappa = 0$. Then S belongs to one of the 4 following cases.

1. $p_g = 0$, $q = 0$: then $2K = 0$, and we say that S is an Enriques surface.
2. $p_g = 0$, $q = 1$: then S is a bielliptic surface.
3. $p_g = 1$, $q = 0$: then $K \equiv 0$, and we say that S is a K3 surface.
4. $p_g = 1$, $q = 2$: then S is an abelian surface.

3.1. **Abelian surfaces.** We've seen these.

3.2. **Bielliptic surfaces. Definition.** A surface is *bielliptic* if $S \cong (E \times F)/G$, where E and F are elliptic curves, and G is a finite group of translations of E acting on F such that $F/G \cong \mathbb{P}^1$. (*Caution:* Used to be called hyperelliptic surfaces.)

Theorem. Then every bielliptic surface is one of the following types:

1. $(E \times F)/G$, $G = \mathbb{Z}/2$ acting on F by $x \mapsto -x$.
2. $(E \times F)/G$, $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ acting on F by $x \mapsto -x$, $x \mapsto x + \epsilon$ ($\epsilon \in F_2$),
3. $(E \times F_i)/G$, $G = \mathbb{Z}/4\mathbb{Z}$ acting on F by $x \mapsto ix$.
4. $(E \times F_i)/G$, $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2$ acting on F by $x \mapsto ix$, $x \mapsto x + (1+i)/2$.
5. $(E \times F_\rho)/G$, $G = \mathbb{Z}/3$ acting by $x \mapsto \rho x$.
6. $(E \times F_\rho)/G$, $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ acting by $x \mapsto \rho x$, $x \mapsto (1-\rho)/3$.
7. $(E \times F_\rho)/G$, $G = \mathbb{Z}/6$ acting by $x \mapsto -\rho x$.

They are always elliptic fibrations. They have covering spaces of degree at most 6 that are abelian surfaces. Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 0 & & 2 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

3.3. **K3 surfaces.** K3 surface: $p_g = 1$, $q = 0$, implies $K \equiv 0$, simply connected.

Examples:

1. complete intersections: quartic in $\mathbb{P}^3$, intersection of cubic and quadric in $\mathbb{P}^4$, and intersection of 3 quadrics in $\mathbb{P}^5$.
2. Double cover of $\mathbb{P}^2$ branched over a sextic curve
3. Let A be an abelian surface, and let τ be the involution given by $a \mapsto -a$. The fixed points are the 2-torsion points, of which there are 16 in $\mathbb{C}^2/\Lambda \cong \mathbb{R}^4/\Lambda$. The quotient will be singular at these 16 points, with singularity analytically isomorphic to $y^2 + x^2 = z^2$ in $\mathbb{C}^3$. Blow these up, to get something smooth; the exceptional

divisors are (-2)-curves. This is a K3 surface, called the *Kummer surface* of A. **Easy Exercise.** Any smooth rational curve on a K3 surface is a (-2)-curve.

Noether's formula

$$\chi(\mathcal{O}_S) = \frac{1}{12}\left(K_S^2 + \chi_{top}(S)\right)$$

gives $\chi_{top} = 24$ from which $b_2 = 22$. Thus the Hodge diamond is:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The algebraic K3 surfaces come in families, of which the first few are on the example list. Given an ample divisor D on a K3 surface, of self-intersection d. Then the K3 along with the divisor is called a "degree d K3 surface". It turns out that d is an even integer, and for each such d the space of degree d K3 surfaces is irreducible of dimension 19, and determined by the Hodge structure on H^2. (I believe this is due to Piatetski-Shapiro. This is one of th most famous examples of a *Torelli theorem.*)

For example: $d = 4$, get the quartic surfaces. (**Exercise:** check that there is a 19-dimensional family of such K3's.) $d = 6$, $d = 8$ also get complete intersections described above. (**Same exercise.**) $d = 2$, get the double covers of $\mathbb{P}^2$. There is a 19-dimensional family of them: There is a $\binom{8}{2} = 28$-dimensional family of degree 6 equations in 3 variables, minus 1 to projectivize, minus $\dim \operatorname{Aut} \mathbb{P}^2 = 8$ for automorphisms of $\mathbb{P}^2$, so we indeed get 19.

3.4. **Enriques surfaces.** Enriques surfaces are K3 surfaces, quotient by a fixed-free involution.

Example: Let S be the quartic K3 surface in $\mathbb{P}^3$ defined by $x^4 + y^4 = z^4 + w^4$. Let σ be the automorphism sending $[x : y : z : w]$ to $[x; iy; -z; -iw]$, which has order 4, and acts on S. The quotient of S by the the cyclic group generated by σ is an algebraic surface X. $K \neq 0$, but $2K = 0$. (Why didn't I mod out by σ^2? Because that has a fixed point!)

Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Enriques surfaces are always elliptic fibrations.

That completes the case of Kodaira dimension 0.

4. Kodaira dimension 1 and elliptic surfaces

Theorem. Any surface of Kodaira dimension 1 is an elliptic surface.

In fact some elliptic surfaces have lower Kodaira dimension, e.g. Enriques surfaces and $E_1 \times E_2$. They never have higher.

5. Surfaces of general type

OMIT THIS IF THERE ISN'T ENOUGH TIME.

Lemma. If $K^2 > 0$, then there is an integer n_0 such that ϕ_{nK} maps S birationally onto its image for all $n \geq n_0$.

Proof. Let H be a hyperplane section of S. Since $K^2 > 0$, Riemann-Roch gives us:

$$h^0(nK - H) + h^0(H + (1-n)K) \to \infty$$

as $n \to \infty$. We have $H \cdot K > 0$ as S is non-ruled (omitted), hence $(H + (1-n)K) \cdot H < 0$ for n sufficiently large. Thus there is an n_0 such that $h^0(nK - H) \geq 1$ for all $n \geq n_0$. Let $E \in |nK - H|$. It is clear that the system $|nK| = |H + E|$ separates points of $S - E$, and separates tangents to points of $S - E$. The restriction of ϕ_{nK} to $S - E$ is thus an embedding. □

Proposition. Let S be a minimal surface. Then the following are equivalent.

1. $\kappa = 2$.
2. $K^2 > 0$ and S is irrational.
3. there exists an integer n_0 such that ϕ_{nK} is a birational map of S to its image for $n \geq n_0$.

Two-thirds of the proof are easy.

5.1. Inequalities among various invariants.

Noether's inequality: $p_g \leq 2 + K^2/2$.

This implies $K^2 \geq 2\chi(\mathcal{O}_X) - 6$. *Proof.* $\chi(\mathcal{O}_S) \leq 1 + p_g$, from which $2\chi(\mathcal{O}_S) - 6 \leq 2p_g - 4 \leq K^2$. □

Theorem. Let S be a non-ruled surface. Then $\chi_{top}(S) \geq 0$ and $\chi(\mathcal{O}_S) \geq 0$. Moreover, if S is of general type, then $\chi(\mathcal{O}_S) > 0$.

Immediately: $p_g \geq q$.

Also, substituting $\chi_{top} \geq 0$ into Noether's equality

$$\chi(\mathcal{O}_S) = \frac{1}{12}\left(K_S^2 + \chi_{top}(S)\right)$$

gives $K^2 \leq 12\chi(\mathcal{O}_S)$.

This last inequality can be strengthened: *Bogomolov-Miyaoka-Yau inequality* $K^2 < 9\chi(\mathcal{O}_S)$. (Bogomolov is the next speaker in the algebraic geometry seminar.)

Geography: $K^2 \geq 1$, $\chi(\mathcal{O}_S) \geq 1$, Noether $K^2 \geq 2\chi(\mathcal{O}_S) - 6$, BMY $K^2 \leq 9\chi(\mathcal{O}_S)$. Region in the first quadrant of the $(\chi(\mathcal{O}_S), K^2)$-plane. Which are achievable? Persson's theorem: all values with $K^2 \leq 8\chi(\mathcal{O}_S)$ occur.

Note that the signature of the intersection form is $\tau = (K^2 - 2\chi_{top})/3 = (K^2 - 2(12\chi(\mathcal{O}_S) - K^2))/3 = K^2 - 8\chi(\mathcal{O}_S)$, so Persson's surfaces all have negative signature. Surfaces with positive signature are in general much harder to construct.

Hard to construct things on the BMY line.

6. Conclusion of the course

My subtext was to teach you enough algebraic geometry so you could use it and read it in your working life. We've seen most of the tools that you'll need to be familiar with to read much of the literature; you're now equipped to ready the rest of Beauville or Barth-Peters-Van de Ven. If you have any follow-up questions, as always just drop by.