COMPLEX ALGEBRAIC SURFACES CLASS 18

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CONTENTS

Today, I'd like to prove one last general fact about surfaces:

Theorem. Let S, S' be two non-ruled minimal surfaces. Then every birational map from $S' \dashrightarrow S$ is an isomorphism. In particular, every non-ruled surface admits a unique minimal model (up to isomorphism); the group of birational maps from a non-ruled minimal surface to itself coincides with the group of automorphisms of the surface.

It makes use of the last general construction commonly used in the theory of algebraic surfaces, the Albanese variety. After seeing this, I think you'll be fully equipped to read much of the literature on surfaces in the algebraic category. On Friday, the last day of class, I'll then sketch the rest of the classification of algebraic surfaces. In particular, you're familiar with all the ingredients of the proof.

Last week we started to talk about:

1. ALBANESE VARIETY

1.1. The universal property of the Albanese variety.

An *abelian variety* over $\mathbb{C}$ is an projective algebraic variety that is a complex torus $\mathbb{C}^n/L$ (L a lattice of rank $2n$); it is a group.

In the algebraic category, an abelian variety A over a field k is a projective group variety, i.e. a variety such that the multiplication map $A \times A \to A$ and the inverse map $A \to A$ are both algebraic morphisms. *Facts:* (1) A is smooth, and the group is necessarily abelian. (2) Any map between two abelian varieties sending 0 to 0 is necessarily a group homomorphism.

Date: Wednesday, December 4.

Lemma. The m-forms on the complex torus $\mathbb{C}^n/L$, where C^n has coordinates $x_1, \dots, x_n$, are precisely those of the form $\sum_{i_1<i_2<\dots<i_m}(\text{constant})dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_m}$.

Proof. If you have an m form on a complex torus, lift it to an m-form on $\mathbb{C}^n/L$; you get an m-form there that looks like: $\sum_{i_1<i_2<\dots<i_m}(\text{hol. function})dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_m}$. That holomorphic function is period, and hence by the maximum principle must be constant. □

The analogous fact is true for abelian varieties: if $p \in A$, the m-forms are canonically identified with $\wedge^m T_p^*$.

Hard theorem. For any (smooth projective) variety X over a field k, there exists an abelian variety $\mathrm{Alb}(X)$ and a morphism $\alpha_X : X \to \mathrm{Alb}(X)$ with the following universal property: *for any abelian variety T and any morphism $f : X \to T$, there exists a unique morphism (up to translation) $\tilde{f} : A \to T$ such that $\tilde{f} \circ \alpha = f$.*

Exercise. A is determined up to isomorphism.

Here's a way to think of this. If you have a set S, then there is a group generated by this set, called the free group F_S, along with a map of sets $\alpha_S : S \to F_S$. It satisfies the universal property such that if G is any group, and $f : S \to G$ is any map of sets, then f factors uniquely through α_S:

$$S \stackrel{\alpha_S}{\to} F_S \stackrel{g}{\to} G$$

where g is a group homomorphism. Then you can check without any machinery that if $S \stackrel{\alpha_S}{\to} F_S$ exists, then it is unique up to unique isomorphism. Then you still have to prove existence.

Similarly, after cutting and pasting, we have: If you have a smooth projective variety S, then there is an "abelian variety generated by S", called the Albanese variety $\mathrm{Alb}(S)$, along with a map of varieties $\alpha_S : S \to \mathrm{Alb}(S)$. It satisfies the universal property such that if A is any abelian variety, and $f : S \to A$ is any map of varieties, then f factors uniquely (up to translation in the target) through α_S:

$$S \stackrel{\alpha_S}{\to} \mathrm{Alb}(S) \stackrel{g}{\to} A$$

where g is a morphism of abelian varieties. Then you can check without any machinery that if $S \stackrel{\alpha_S}{\to} \mathrm{Alb}(S)$ exists, then it is unique up to unique isomorphism. Then you still have to prove existence.

Hard theorem continued. α induces an isomorphism $\alpha^* : H^0(A, \Omega^1_A) \to H^0(X, \Omega^1_X)$.

Some fast consequences.

• $\dim \mathrm{Alb}(X) = h^1(X, \Omega^1_X)$. In particular, if X is a surface, $\dim \mathrm{Alb}(X) = q$. So as a corollary, if $q = 0$, every morphism from X to a complex torus is trivial (i.e. every map is reduced to a point). Also, every map from $\mathbb{P}^1$ to a complex torus is trivial. (Explain why in the complex category.)

• The Albanese variety is functorial in nature. If $f : X \to Y$ is a morphism of smooth projective varieties, there is a unique morphism $F : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ \mathrm{Alb}(X) & \xrightarrow{\text{exists unique } F} & \mathrm{Alb}(Y) \end{array}$$

is commutative.

Moreover, the Abelian variety $\mathrm{Alb}(X)$ is generated by $\alpha(X)$ (i.e. there is no abelian subvariety of $\mathrm{Alb}(X)$ containing $\alpha(X)$), as the Abelian subvariety of $\mathrm{Alb}(X)$ generated by $\alpha(X)$ satisfies the universal property. In particular, $\alpha(X)$ is not reduced to a point if $\mathrm{Alb}(X)$ is not a point.

• Hence if f is surjective, then so is the morphism $F : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$. Reason: Let A be the image of $\mathrm{Alb}(X)$ in $\mathrm{Alb}(Y)$, so $\alpha_Y(Y) \subset A \subset \mathrm{Alb}(Y)$. As $\alpha_Y(Y)$ spans $\mathrm{Alb}(Y)$, we must have $A = \mathrm{Alb}(Y)$. (This is true for free groups too!)

• Suppose α_X factors as $S \xrightarrow{f} T \to \mathrm{Alb}(S)$, with f surjective. Then the induced morphism $\tilde{j} : \mathrm{Alb}(T) \to \mathrm{Alb}(S)$ is an isomorphism.

Proof. Consider

$$\begin{array}{ccccc} S & \xrightarrow{f} & T & & \\ \downarrow \alpha_S & & \downarrow \alpha_T & \searrow & \\ \mathrm{Alb}(S) & \xrightarrow{F} & \mathrm{Alb}(T) & \xrightarrow{\tilde{j}} & \mathrm{Alb}(S) \end{array}$$

The composition of the morphisms on the bottom row is the identity by the universal property. F is surjective. □

• If C is a (smooth projective) curve, we have a candidate for the Albanese variety, given by $C \to \mathrm{Pic}^0(C)$. ($\mathrm{Pic}^0(C)$ is an abelian variety, often written $\mathrm{Jac}(C)$.) Choose a p_0. Send p to $\mathcal{O}_C(p - p_0)$. If $g(C) > 0$, then this is an injection of sets: $\mathcal{O}_C(p - p_0) \cong \mathcal{O}_C(q - q_0)$ implies $\mathcal{O}_C(p) \cong \mathcal{O}_C(q)$, implies we have a degree 1 map to $\mathbb{P}^1$, implies $C \cong \mathbb{P}^1$.

$\mathrm{Jac}(C)$ has dimension $g(C)$ (done earlier).

Fact: If $g(C) > 0$, then $C \to \mathrm{Jac}(C)$ is a closed immersion.

Fact: The Jacobian satisfies the universal property, i.e. $\mathrm{Alb}(X) = \mathrm{Jac}(C)$.

1.2. **More serious facts.** Suppose you have a map from a smooth surface S to a projective, but possibly singular, curve C. Then there is a new curve $Norm(C)$ along with a map $Norm(C) \to C$, called the *normalization* of C. It is a desingularization of C.

Normalization fact: Any map from S to C factors through $Norm(C)$. (You may find this believable.)

This is a special case of the following much more general fact, that is not too hard. (We won't use it.)

There is a normalization operation $\tilde{X} \to X$ for an arbitrary variety X (that roughly corresponds to integral closure of the ring of functions). It is a partial desingularization in general. A variety is *normal* if it is equal to its normalization. In particular, a smooth variety is normal. Then any map from a normal variety X to a variety Y factors through $Norm(Y)$.

The proof is short; if you'd like to see it, just ask. (The hardest thing, surprisingly enough, is showing that a smooth variety is normal.)

Fact: Stein factorization (hard, but useful in many circumstances). Suppose $f : X \to Y$ is a proper morphism of varieties. (Draw picture.) Then f factors as $X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$, where h is a finite morphism, and g has connected fibers.

Put on side board:

Proposition. Let S be a surface, and suppose that $\alpha(S)$ is a curve C (not necessarily smooth). Then C is a smooth curve of genus q, and the fibres of α are connected.

Proof. We know that $q > 0$.

Since S is normal, $S \to C$ factors through $S \to Norm(C) \to C \to \mathrm{Alb}(S)$. So $\mathrm{Jac}(Norm(C)) \to \mathrm{Alb}(S)$ is an isomorphism. Thus $Norm(C)$ has genus q, and as $Norm(C) \to C \to \mathrm{Jac}(Norm(C))$ is an embedding, we must have $Norm(C) = C$ so C is a smooth curve of genus q.

By Stein factorization, we can factorize α_S as $S \to \tilde{C} \to C$ where the first morphism has connected fibers, and the second is finite. Factor further: $S \to Norm(\tilde{C}) \to \tilde{C} \to C \to \mathrm{Alb}(S)$. As before, we have an isomorphism of $\mathrm{Alb}(S)$ with $\mathrm{Jac}(Norm(\tilde{C}))$ as well as with $\mathrm{Jac}(C)$, from which $Norm(\tilde{C}) \to C$ is an isomorphism, and that big factorization now collapses to

$$S \overset{\text{conn. fibres}}{\to} C \hookrightarrow \mathrm{Alb}(S).$$

□

2. Proof of uniqueness of minimal model of non-ruled surfaces.

Put on side board:

Lemma Let S be a surface with $p_g = 0$, $q \geq 1$. Then $\alpha_S(S)$ is a curve. (This is a characteristic 0 proof; I don't know if it is known if this is true in positive characteristic!)

Proof. (Complex proof; but with word changes, it works in the algebraic category in characteristic 0.)

Note that $p_g = 0$ means that there are no non-zero 2-forms on S.

If $\alpha_S(S)$ is a point, then $q = 0$, contradiction. So assume otherwise that $\alpha_S(S)$ is a surface. Then there is some point $p \in \alpha(S)$ near which the morphism $S \to \alpha_S(S)$ is a covering space, i.e. an analytic-local isomorphism. (Algebraic word: etale. Here is where we use the characteristic 0 hypothesis.) Take local coordinates $u_1, \ldots, u_q$ such that $\alpha_S(S)$ is cut out by $u_3 = u_4 = \cdots = u_q = 0$. Then $du_1 \wedge du_2 \in \wedge^2 T_p^*(\mathrm{Alb}(S))$ is a 2-form at p, which extends to a non-zero 2-form on $\mathrm{Alb}(S)$. Pull this back to S to get a 2-form on S. This is non-zero at the preimage of p, contradicting $p_g = 0$. $\square$

2.1. **Proof of the theorem.** *Step 1.* Suppose ϕ is some birational transformation $S' \dashrightarrow S$ violating the statement. Then by resolution of indeterminacy, we can resolve the morphism to by blowing up S' a minimal number of times such that the induced rational map $f : S' \dashrightarrow S$ is a morphism; this minimal number is positive, or else ϕ would have been a morphism to start with. Let E be the exceptional curve of the last blow-up. Then $f(E)$ is a curve C on S, as otherwise $f(E)$ is a point, and that last blow-up was redundant.

Step 2. We can factor f' into a number of blow-ups as well. I claim that $E \cdot K_{S'} \geq C \cdot K_S$, with equality iff C doesn't contain any points in any of these new blow-ups. To show this, I can do it one blow-up at a time. Suppose we blow-up a point p on C, to get $Bl : S'' \to S$, with exceptional divisor E″. Then recall that $K_{S''} = Bl^* K_S + E''$. Also, $C^{\text{strict}} = Bl^* C - m_{p,C} E''$. Thus

$$C^{\text{strict}} \cdot K_{S''} = (Bl^* C - m_{p,C} E) \cdot (Bl^* K_S + E'') = C \cdot K_S - m_{p_C}.$$

Hence at every stage, the value of curve dot canonical can't increase: it stays the same if we blow up a point away from the curve, and it strictly decreases if we blow up a point on the curve.

Now $E^2 = -1$, $g(E) = 0$, so by the genus formula $2g(E) - 2 = E^2 + K_{S'} \cdot E$ we have $K_S \cdot E = -1$.

Thus $K_S \cdot C \leq -1$. By the genus formula for C, we have

$$2g(C) - 2 = C^2 + K_S \cdot C$$

from which $C^2 \geq -1$, with equality iff $g(C) = 0$ iff C is smooth rational curve with self-intersection -1. This is impossible as S is minimal, so we conclude:

$$C^2 \geq 0, K_S \cdot C \leq -1.$$

Step 3. Remarkably, this implies that *all plurigenera* P_n vanish! I.e. $h^0(S, K_S^{\otimes n}) = 0$. Otherwise, if there were a section of nK_S, then by our useful lemma $(nK_S) \cdot C \geq 0$, contradiction.

In particular, $p_g = 0$ (aka P_1), and $P_2 = 0$.

Step 4. If $q = 0$, then Castelnuovo's rationality criterion says that S is rational, contradicting the hypothesis. So $q > 0$.

Step 5. By our lemma, $\alpha_S(S)$ is a curve B of genus $q > 0$. Moreover, $\alpha_S(S) : S \to B$ has connected fibers.

C is rational, so C is contained in a fiber of this map, say F. Since $C^2 \geq 0$, we must have $F = rC$. Thus $C^2 = 0$, hence $C \cdot K = -2$. By the genus formula, if F' is a general fiber, then

$$2g(F') - 2 = F'(K + F') = -2r$$

from which $r = 1$ and $g(F) = 0$. Then the Noether-Enriques theorem implies that S is ruled, contradiction. $\square$