COMPLEX ALGEBRAIC SURFACES CLASS 17

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On board beforehand:

- Useful trick. $|D| \neq \emptyset$ (i.e. $h^0(D) > 0$), C irreducible, $C^2 \geq 0$ implies $DC \geq 0$.
- Genus formula. $2g(C) - 2 = C \cdot (K_S + C)$.
- Riemann-Roch: $\chi(D) = \chi(\mathcal{O}) + \frac{1}{2} D \cdot (D - K)$.
- Riemann-Roch: In the case when $h^1(\mathcal{O}) = 0$ and $h^0(K - D) = 0$, we have $h^0(D) \geq 1 + \frac{1}{2} D \cdot (D - K)$ (with equality iff $h^1(D) = h^2(\mathcal{O}) = 0$).

Also, I have been repeatedly using a fact that I may not have explicitly mentioned. A singular curve C still has a notion of genus (called the *arithmetic genus*), $g(C) = 1 - \chi(\mathcal{O}_C)$, which is $h^1(C, \mathcal{O}_C)$ if C is connected. (Warning: this is usually denoted p_a.) Fact: C irreducible and $g(C) = 0$ implies that C is smooth and genus 0.

1. CASTELNUOVO'S THEOREM

We spent last time on most of the proof of :

Theorem: Castelnuovo's Rationality Criterion. Let S be a surface with $q = P_2 = 0$. Then S is rational.

Reminder. $q = h^1(S, \mathcal{O}_S) = h^0(S, \Omega_S) = h^2(S, \Omega_S) = h^1(S, K_S)$ (draw Hodge diamond). This is called the *irregularity* of a surface. $P_2 = h^0(S, K_S^{\otimes 2})$.

We first reduced this to:

Castelnuovo'. Let S be a minimal surface with $q = P_2 = 0$. Then there exists a smooth rational curve C on S such that $C^2 \geq 0$.

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And then reduced it to:

Castelnuovo". $q = P_2 = 0$ and S minimal implies that there is an effective divisor E on S (i.e. $|E| \neq \emptyset$) such that $K \cdot E < 0$ and $|K + E| = \emptyset$. *Keep on board.*

The proof was in 3 cases: $K^2 < 0, = 0$, and > 0.

We did the cases $K^2 = 0$ and $K^2 > 0$, so we're left with:

Proof of Castelnuovo" in the case $K^2 < 0$.

Step 1. Find an effective divisor E on S such that $K \cdot E < 0$. (2 out of the 3 desired conclusions.) Let H be a hyperplane section of S. If $K \cdot H < 0$, we can take $E = H$, and we win. If $K \cdot H = 0$, then $K + nH$ has sections for $n \gg 0$, and we win again: take $E \in |K + nH|$; it is effective, and $K \cdot E < 0$.

So now assume $K \cdot H > 0$. Let $r_0 = K \cdot H/(-K^2)$. This is positive. Note that

$$(H+r_0K)^2 = H^2+2(K\cdot H/(-K^2))HK+(K\cdot H)^2/(-K^2)^2K^2 = H^2+(K\cdot H)^2/(-K^2) > 0.$$

(Don't be nervous with rational coefficients; we can still formally manipulate them in the same way. We should just be careful about them being actual divisors.) Also $(H + r_0K) \cdot K = 0$. So if p/q ($p, q \in \mathbb{Z}^+$) is a touch greater than r_0, we have

$$(H + \frac{p}{q}K)^2 > 0, \quad (H + \frac{p}{q}K) \cdot K < 0, \quad (H + \frac{p}{q}K) \cdot H > 0.$$

Let $D_m = mq(H + \frac{p}{q}K)$, so

$$D_m^2 > 0, \quad D_M \cdot K < 0, \quad D_m \cdot H > 0.$$

As $m \to \infty$, $h^0(K - D_m) = 0$, as $(K - D_m) \cdot H < 0$. By Riemann-Roch,

$$h^0(D_m) \geq \chi(\mathcal{O}_X) + \frac{1}{2}(D_m^2 - D_m \cdot K) \to \infty$$

as $m \to \infty$. Hence $h^0(D_m) > 0$ for some large m; take $E \in |D_m|$. It is effective, and $K \cdot E < 0$.

Step 1. Remember our *useful remark* from a few classes ago: If D is an effective divisor, and C is an irreducible curve such that $C^2 \geq 0$, then $D \cdot C \geq 0$.

Suppose we have an effective divisor E on S such that $K \cdot E < 0$. Then let C be some irreducible component of E such that $K \cdot C < 0$. By the genus formula,

$$2g(C) - 2 = K \cdot C + C^2 \Rightarrow -2 = K \cdot C + C^2 - 2g(C) \leq -1 + C^2,$$

so $C^2 \geq -1$, and $C^2 = -1$ iff C is an exceptional curve, which is excluded. Hence $C^2 \geq 0$.

Thus $(aC + nK) \cdot C < 0$ for $n \gg 0$. Thus by the useful remark, $h^0(S, aC + nK) = 0$ for $n \gg 0$. Clearly $h^0(S, aC+0K) > 0$ (as there is a divisor in the linear system aC, notably aC itself), so there is some n such that $|aC + nK| \neq \emptyset$, $|aC + (n+1)K| = \emptyset$. If $D \in |aC + nK|$, then $K \cdot D \leq -a$, and $|K + D| = \emptyset$. $\square$

2. Application: The only minimal rational surfaces are $\mathbb{P}^2$ and $\mathbb{F}_n$ $(n \neq 1)$

Suppose S is a minimal rational surface. Let H be a hyperplane section, i.e. a very ample divisor. Consider the set A of smooth rational curves C with $C^2 \geq 0$. By Castelnuovo', this set is non-empty.

Castelnuovo'. Let S be a minimal surface with $q = P_2 = 0$. Then there exists a smooth rational curve C on S such that $C^2 \geq 0$.

Keep on board. Let m be the smallest possible choice of C^2 in this family. Among those with this value of C^2, choose some C with the smallest possible value of $C \cdot H$ (which is necessarily positive).

Lemma. All curves in $|C|$ are smooth and genus 0.

Proof. Suppose $D = \sum n_i C_i \in |C|$ isn't irreducible. $K \cdot D < 0$, so choose some C_j such that $K \cdot C_j < 0$. I'll show you that C_j is a smooth rational curve with $C_j^2 \leq C^2$ and $H \cdot C_j < H \cdot C$, contradicting our choice of C.

Andre pointed out a gap here in the original notes. Here is the patch. By the useful trick, if $h^0(S, K_S + C) > 0$, then $(K_S + C) \cdot C \geq 0$. But by the genus formula, $(K_S + C) \cdot C = -2$.

Recall:

Exercise. (I proved this when showing that Castelnuovo" implies Castelnuovo', but it's a good exercise to re-prove it yourself.) If $K_S \cdot C < 0$ and $h^0(S, K_S + C) = 0$ and $q = 0$, then C is a smooth rational curve with $C^2 \geq -1$. Method: apply Riemann-Roch to $K_S + C$.

Now here $q = 0$ (as S is rational), and $h^0(S, \mathcal{O}_S(K + C_j)) \leq h^0(S, \mathcal{O}_S(K + \sum n_i C_i)) = 0$. So C_j is a smooth rational curve. Also, $C^2 = (\sum n_i C_i)^2 \geq \sum n_i^2 C_i^2 \geq C_j^2$. Finally, $C \cdot H = (\sum n_i C_i) \cdot H \geq C_j \dot{H}$. □

Lemma. $\dim |C| \leq 2$, i.e. $h^0(S, \mathcal{O}(C)) \leq 3$.

Proof. Assume otherwise. Pick some point $p \in S$. Pick an open subset containing p, over which $\mathcal{O}(C)$ is trivialized. Then via this trivialization, the $h^0(S, \mathcal{O}(C))$ corresponds to some vector space functions, of dimension at least 4. Thus there is some non-zero function that restricts to something vanishing to order 2 at p. In that case, the zero set is singular, contradicting our previous lemma. □

Lemma. $m = 0$ or 1.

Proof. We know $m \geq 0$ by construction. Let C_0 be a fixed curve in that class. Consider the exact sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_{C_0}(m) \to 0$. As $q = 0$, this is exact on global sections, so $3 \geq h^0(S, \mathcal{O}_S(C)) = 1 + h^0(\mathbb{P}^1, \mathcal{O}(m)) = 2 + m$. □

Note also that $\mathcal{O}(C)$ has no base points. (Explain.)

Theorem. Let S be a minimal rational surface. Then $S \cong \mathbb{P}^2$ or some $\mathbb{F}_n$ for $n \neq 1$.

Proof. If $m = 0$, then $|C|$ determines a morphism $S \to \mathbb{P}^1$ (as there are no base points), all of whose fibres are smooth rational curves. So S is a rational ruled surface $\mathbb{F}_n$.

If $m = 1$, then $|C|$ determines a morphism $S \to \mathbb{P}^2$. The preimages are reduced points (any two sections intersect in $m = 1$ point, counted with multiplicity), so this is an isomorphism. $\square$

3. Albanese variety

I next want to introduce the last important tool you haven't yet seen in the classification of surfaces, the *albanese variety*.

As a goal, I will prove (next day):

Theorem. Let S, S' be two non-ruled minimal surfaces. Then every birational map from S' to S is an isomorphism. In particular, every non-ruled surface admits a unique minimal model (up to isomorphism); the group of birational maps from a non-ruled minimal surface to itself coincides with the group of automorphisms of the surface.

3.1. **The universal property of the Albanese variety.**

An *abelian variety* over $\mathbb{C}$ is an projective algebraic variety that is a complex torus $\mathbb{C}^n/L$ (L a lattice of rank $2n$); it is a group. In the purely algebraic category, an abelian variety over a field k is a smooth projective group scheme.

Hard theorem. For any (smooth projective) variety X over a field k, there exists an abelian variety A and a morphism $X \to A$ with the following universal property: *for any abelian variety T and any morphism $f : X \to T$, there exists a unique morphism $\tilde{f} : A \to T$ such that $\tilde{f} \circ \alpha = f$.*

Exercise. A is determined up to isomorphism.

This is called the *Albanese variety* of X, written $\mathrm{Alb}(X)$.

Hard theorem continued. α induces an isomorphism $\alpha^* : H^0(A, \Omega^1_A) \to H^0(X, \Omega^1_X)$.

The proof in general is hard, but over $\mathbb{C}$ there is an easier proof using Hodge theory (see Beauville, for example). For simplicity, I'll work in the complex category.

Some fast consequences.

$\dim \mathrm{Alb}(X) = h^1(X, \Omega^1_X)$. In particular, if X is a surface, $\dim \mathrm{Alb}(X) = q$. So as a corollary, if $q = 0$, every morphism from X to a complex torus is trivial (i.e. every map is reduced to a point). Also, every map from $\mathbb{P}^1$ to a complex torus is trivial. (Explain why in the complex category.)

The Albanese variety is functorial in nature. If $f : X \to Y$ is a morphism of smooth projective varieties, there is a unique morphism $F : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ \mathrm{Alb}(X) & \xrightarrow{\text{exists unique } F} & \mathrm{Alb}(Y) \end{array}$$

is commutative.

Moreover, the Abelian variety $\mathrm{Alb}(X)$ is generated by $\alpha(X)$ (i.e. there is no abelian subvariety of $\mathrm{Alb}(X)$ containing $\alpha(X)$), as the Abelian subvariety of $\mathrm{Alb}(X)$ generated by $\alpha(X)$ satisfies the universal property. In particular, $\alpha(X)$ is not reduced to a point if $\mathrm{Alb}(X)$ is not a point. It also follows that if the morphism f is surjective, then so is the morphism $F : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$.

Fact: If X is a curve, then $\mathrm{Alb}(X)$ is the Jacobian $\mathrm{Pic}^0(X)$ (usually denoted $\mathrm{Jac}(X)$).