COMPLEX ALGEBRAIC SURFACES CLASS 16

RAVI VAKIL

CONTENTS

1. Castelnuovo's Theorem 1
1.1. Motivation: Minimal rational surfaces 1
1.2. Motivation Luroth's theorem (in characteristic 0) 2
2. Proof of Castelnuovo's criterion (part 1) 2

On board beforehand:

- Useful trick. $|D| \neq \emptyset$ (i.e. $h^0(D) > 0$), C irreducible, $C^2 \geq 0$ implies $DC \geq 0$.
- Genus formula. $2g(C) - 2 = C \cdot (K_S + C)$.
- Riemann-Roch: $\chi(D) = \chi(\mathcal{O}) + \frac{1}{2} D \cdot (D - K)$.
- Riemann-Roch: In the case when $h^1(\mathcal{O}) = 0$ and $h^0(K - D) = 0$, we have $h^0(D) \geq 1 + \frac{1}{2} D \cdot (D - K)$ (with equality iff $h^1(D) = h^2(\mathcal{O}) = 0$).

1. CASTELNUOVO'S THEOREM

We saw how tricky it was to show that a surface is rational.

Theorem: Castelnuovo's Rationality Criterion. Let S be a surface with $q = P_2 = 0$. Then S is rational.

Reminder. $q = h^1(S, \mathcal{O}_S) = h^0(S, \Omega_S) = h^2(S, \Omega_S) = h^1(S, K_S)$ (draw Hodge diamond). This is called the *irregularity* of a surface.

$P_2 = h^0(S, K_S^{\otimes 2})$.

It was once believed that this could be weakened to $q = P_1 = 0$, which is somehow more attractive (as P_1 is an entry in the Hodge diamond), but this false, and we may see examples before the end of the course (Enriques surfaces, Godeaux surfaces).

1.1. Motivation: Minimal rational surfaces.

We know lots of rational surfaces now: $\mathbb{P}^2$, $\mathbb{F}_n$, and blow-ups of these. At this point, we may suspect that we've found them all. How can we show this? We'll use Castelnuovo's criterion.

Date: Friday, November 22.

1.2. **Motivation Luroth's theorem (in characteristic 0).** A variety V of dimension n is *unirational* if there is a dominant map (i.e. one with dense image) $\mathbb{P}^n \dashrightarrow V$.

Lüroth's Theorem. Every unirational curve is rational.

Proof. This is true in arbitrary characteristic, but here's a proof that works only in characteristic 0. Suppose $\mathbb{P}^1 \dashrightarrow C$, where C is a curve, possibly singular and not proper. Then we also get a rational map $\mathbb{P}^1 \dashrightarrow C'$, where C' is a smooth compactification of a smoothing of C. By our lemma from long ago, any rational map from a smooth curve to anything projective extends to a morphism, so we have $\mathbb{P}^1 \to C'$. Dominant implies surjective. So we can apply the Riemann-Hurwitz formula, to see that

$$2 - 2g(\mathbb{P}^1) = d(2 - 2g(C')) - \text{ramification contribution}.$$

The left side is 2, but if $g(C') > 0$ the right side can't be positive. □

Theorem. In characteristic 0, every unirational surface is rational.

In positive characteristic, the theorem is *false!* Ask Ted Hwa for an example.

Question: where does the following argument break down in positive characteristic?

Proof. Suppose S is a unirational surface. If there was any doubt, let's say that it is smooth and compact. (Otherwise, there is a way of producing a smooth and compact birational model.) So we have $\mathbb{P}^2 \dashrightarrow S$. By the elimination of indeterminacy, we can blow up $\mathbb{P}^2$ and get a morphism $\mathrm{Bl}\,\mathbb{P}^2 \to S$. This morphism is dominant and hence surjective. Interpret $q(S)$ as $H^0(S, \Omega_S)$, and recall $P_2(S) = H^0(S, \mathcal{K}_S^{\otimes 2})$. If $q > 0$ or $P_2 > 0$, then pullback the nonzero form (i.e. section of either Ω_S or $\mathcal{K}_S^{\otimes 2}$) to get a non-zero section of the corresponding bundle on $\mathrm{Bl}(\mathbb{P}^2)$. This would give $q(\mathrm{Bl}(\mathbb{P}^2)) > 0$ or $P_2(\mathrm{Bl}(\mathbb{P}^2)) > 0$.

Hence $q(S) = P_2(S) = 0$. Then by Castelnuovo, S is rational. □

Remark. Even in characteristic 0, there are 3-folds that are unirational but not rational, and they are not even that exotic! It is not hard to show that smooth cubic threefolds in $\mathbb{P}^4$ are all unirational; Clemens and Griffiths showed that *none* of them are rational! Iskovskih and Manin did the same for quartic threefolds as well.

2. Proof of Castelnuovo's criterion (part 1)

We'll make a couple of reduction steps.

Castelnuovo'. Let S be a minimal surface with $q = P_2 = 0$. Then there exists a smooth rational curve C on S such that $C^2 \geq 0$. *Keep on board.*

Proof that Castelnuovo' implies Castelnuovo's criterion.

$\mathcal{O}_S(C)$ clearly has a section, one whose zero set is C. We'll see that in fact $h^0(S, \mathcal{O}_S(C)) \geq 2$, so "the curve moves". Consider $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$. Now $q = h^1(S, \mathcal{O}_S) =$

0, so when we take global sections, the sequence remains exact, so

$$\begin{aligned} h^0(S, \mathcal{O}_S(C)) &= h^0(S, \mathcal{O}_S) + h^0(C, \mathcal{O}_C(C))) \\ &= 1 + C^2 - g(C) + 1 + h^1(C, \mathcal{O}_C(C)) \\ &= 2 + C^2 \qquad \text{(as } C \cong \mathbb{P}^1\text{, and } \mathcal{O}_C(C) \text{ has positive degree)} \\ &\geq 2 \end{aligned}$$

So taking 2 sections, C and one other, we get a rational map $S \dashrightarrow \mathbb{P}^1$. After blowing up, this becomes a morphism $\tilde{S} \dashrightarrow \mathbb{P}^1$. One of its fibers is isomorphic to C. By the Noether-Enriques theorem, it follows that S is rational. □

So now we want to prove Castelnuovo'. Instead we'll prove

Castelnuovo". $q = P_2 = 0$ implies that there is an effective divisor E on S such that $K \cdot E < 0$ and $|K + E| = \emptyset$. *Keep on board: We seek* $|E| \neq$, $|E + K| = \emptyset$, $K \cdot E < 0$.

Castelnuovo" implies Castelnuovo'. For then some component C of E satisfies $K \cdot C < 0$, and any component satisfies $h^0(S, K + C) = 0$. Applying Riemann-Roch to $K + C$ we get

$$\begin{aligned} 0 &= h^0(K + C) \\ &\geq h^0(K + C) - h^1(K + C) + h^0(-C) \\ &= \chi(K + C) \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}((K + C) - K) \cdot (K + C) \\ &> h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) + \frac{1}{2}(C + K) \cdot C \\ &\geq 1 + \frac{1}{2}(C + K) \cdot C \\ &= g(C). \end{aligned}$$

Hence $g(C) = 0$. $(C + K) \cdot C = -2$, hence $C^2 \geq -1$. If $C^2 = -1$, then C is an exceptional curve, and we hypothesized that there weren't any. So Castelnuovo' follows. □

Proof of Castelnuovo" in the case $K^2 = 0$.

How can we possibly use $P_2 = 0$? Only one reasonable way: Our hypothesis $P_2 = 0$ gives $h^2(-K) = 0$ (Serre duality). Hence by Riemann-Roch (and $q = 0$):

$$h^0(-K) \geq h^0(-K) - h^1(-K) + h^2(-K) = h^0(\mathcal{O}) - h^1(\mathcal{O}) + h^2(\mathcal{O}) + K^2 \geq 1 + K^2.$$

(We'll use this in the $K^2 > 0$ case too.)

So $|-K| \neq \emptyset$. Let H be a hyperplane section of S. Then $H \cdot K < 0$. Note:

- If $n = 0$, then $|H + nK| \neq \emptyset$.
- If $n \gg 0$ then $|H + nK| = \emptyset$ (as $(H + nK) \cdot H < 0$)

Thus there is an $n \geq 0$ such that $|H + nK| \neq \emptyset$, but $|H + (n+1)K| = \emptyset$ as $|H| \neq \emptyset$, and $(H + nK) \cdot H < 0$ for $n \gg 0$). Let D be an element. $|K + D| = \emptyset$, and $K \cdot D = -(-K) \cdot H < 0$. ☐

Proof of Castelnuovo" in the case $K^2 > 0$.

Recall $h^0(-K) = 1 + K^2$, so $h^0(-K) \geq -2$. Suppose $D \in |-K|$.

Three cases:

(1) There is a reducible choice of D, i.e. A, B effective with $A + B \in |-K|$.
(2) $\operatorname{Pic}(C) = \mathbb{Z}K$. (This implies that there is no reducible choice of D (why?), but we don't care.)
(3) All divisors in $|-K|$ irreducible, and $\operatorname{Pic}(C) \neq \mathbb{Z}K$.

Case 1: There is a reducible choice of D, i.e. A, B effective with $A + B \in |-K|$. Then $A \cdot K$ or $B \cdot K < 0$, say the former. Then A is an effective divisor on S such that $A \cdot K < 0$, and $|A + K| = |-B| = \emptyset$.

Case 2: $\operatorname{Pic}(C) = \mathbb{Z}K$. This is the only case where characteristic 0 comes up! From the exact sequence

$$H^1(S, \mathcal{O}_S) \to \operatorname{Pic} S \to H^2(S, \mathbb{Z}) \to H^2(S, \mathcal{O}_S)$$

we have $H^2(S, \mathbb{Z}) \cong \operatorname{Pic} S = \mathbb{Z}K$. Thus $b_2 = 1$. By Poincare duality, the intersection form on $H^2(S, \mathbb{Z})$ is unimodular, so $K^2 = 1$. By Noether's formula,

$$1 = \chi(\mathcal{O}_S) = \frac{1}{12}(K^2 + 2 - 2b_1 + b_2)$$

from which $b_1 = -4$, contradiction.

Case 3: All divisors D in $|-K|$ irreducible and $\operatorname{Pic}(C) \neq \mathbb{Z}K$. Suppose H were an effective divisor. As $|-K| \neq \emptyset$, there exists $n > 0$ such that $|H + nK| \neq \emptyset$ and $|H + (n+1)K| = \emptyset$. If $(H + nK) \cdot K < 0$, we'd be done.

Take an H such that $H + nK \neq 0$. Let $E \in |H + nK|$, $E = \sum n_i C_i$. Then $K \cdot E = -D \cdot E$, and by the useful remark $D \cdot E \geq 0$ since D is irreducible. We are painfully close to being done: we have $K \cdot E \leq 0$, and we want $K \cdot E < 0$!

Thus $K \cdot C_i \leq 0$ for some $C = C_i$. Hence $|K + C| = \emptyset$, from which $0 = h^0(K + C) \geq 1 + \frac{1}{2}(C^2 + CK) = g(C)$. $g(C) = 0$, and $C^2 = -2 - K \cdot C$ (genus formula). We have gained exactly one thing in this paragraph: our divisor C is irreducible, whereas our divisor E was not necessarily. We know that $|C| \neq \emptyset$, $|K + C| = \emptyset$, and $K \cdot C \leq 0$, and we want to show that $K \cdot C < 0$.

So we'll assume $K \cdot C = 0$, and find a contradiction. From the genus formula, $C^2 = -2$. We'll calculate $h^0(-K - C)$. Note that $h^0(2K + C) = h^0(2K + (-D)) \leq h^0(K + C) = 0$.

Thus

$$\begin{aligned} h^0(-K-C) \geq \chi(-K-C) &= \chi(\mathcal{O}_X) + \frac{1}{2}((K+C)^2 + K(K+C)) \\ &= 1 + \frac{1}{2}(C^2 + 3KC + 2K^2) \\ &\geq K^2 \\ &\geq 1 \end{aligned}$$

Since $C^2 = -2$, we have $C \neq -K$, so there exists a nonzero effective divisor A such that $A + C \in |-K|$. This contradicts our hypothesis that $|-K|$ has no reducible divisors.

All that's left is:

Proof of Castelnuovo" in the case $K^2 < 0$.