COMPLEX ALGEBRAIC SURFACES CLASS 15

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CONTENTS

We now know that $\mathrm{Bl}_{6\text{ pts}}\,\mathbb{P}^2$ can be embedded in projective space by its anticanonical line bundle, $3H - E_1 - \cdots - E_6$, corresponding to cubics through the 6 points. (Those 6 points are no 3 on a line, not all on a conic.) The image is a cubic surface.

We know also that any smooth cubic surface is anticanonically embedded (using the adjunction formula). Hence lines in S correspond to (-1)-curves.

We've seen that there are 27 lines on $\mathrm{Bl}_{6\text{ pts}}\,\mathbb{P}^2$, and that their configuration has a beautiful structure related to $W(E_6)$.

By a dimension count, we showed that almost all smooth cubics can be described in this way. More precisely: smooth cubics in $\mathbb{P}^3$ are parametrized by cubic forms in 4 variables modulo scalar multiples, a $\mathbb{P}^{20}$. Inside this $\mathbb{P}^{20}$ there is an open set $SmCub$ parametrizing smooth cubics. (In fact, it is $\mathbb{P}^{20}$ minus a divisor.) We saw by a dimension count that there is a dense (Zariski-)open set of $SmCub$ corresponding to blow-ups of $\mathbb{P}^2$ at 6 points.

Morally speaking, we shouldn't expect any more to be true; we should expect that there are a few smooth cubics that are not expressible in this form. Then we'd have some natural questions, such as: what happens to the 27 lines? What happens to the rational parametrization? But by a miracle:

1. Every smooth cubic has 27 lines, and is the blow-up of $\mathbb{P}^2$ at 6 points

Theorem. Any smooth cubic S is the blow-up of $\mathbb{P}^2$ at 6 points, and in particular has 27 lines in that beautiful configuration.

(Last day, I ended with a sketch of the argument I'll give today.)

Three main steps.

Date: Wednesday, November 20.

1. S contains a line.
2. S contains two skew lines.
3. Use these two skew lines to express S as the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at five points.

A useful tool is $\mathbb{G}(1,3)$, the Grassmannian, which is the parameter space of lines (i.e. $\mathbb{P}^1$'s) in $\mathbb{P}^3$.

Proposition. $\mathbb{G}(1,3)$ is a 4-dimensional irreducible smooth variety, and proper (i.e. compact).

(Some discussion as to why it is believable.)

Proof. I'll show all but properness. We build it up as follows

$$\{2 \text{ distinct points in } \mathbb{P}^3\} \leftrightarrow \{(\text{line, 2 distinct points on line})\} \to \mathbb{G}(1,3) = \{\text{lines }\}.$$

The first two are clearly the same variety. The first has dimension 6, and is smooth and irreducible (it is $\mathbb{P}^3 \times \mathbb{P}^3$ minus the diagonal). The morphism from the second to the third is smooth (it has smooth fibers), and the fiber dimension is clearly 2. Thus the third is smooth and irreducible, and has dimension 4. $\square$

Proposition. Every cubic surface (even singular ones) contains a line.

Proof. Consider

$$\{\text{cubics}\}^{19} \leftrightarrow \{(\text{cubic, line in cubic})\}^{19} \xrightarrow{15} \mathbb{G}(1,3) = \{\text{lines }\}^4.$$

Build it from right to left. The third is proper and smooth. The fiber dimension from the second to the third is 15, and the fibers are smooth. To check that the fibers are smooth: given a line in 4-space with coordinates X, Y, Z, T, say defined by $Z = T = 0$, the cubics containing the line correspond to

$$A_1(Z,T)X^2 + 2B_1(Z,T)XY + C_1(Z,T)Y^2 + 2D_2(Z,T)X + 2E_2(Z,T)Y + F_3(Z,T) = 0,$$

where $A_1(Z,T)$ is homogeneous of degree 1, etc. (If the characteristic is 2, then ignore the 2's!) A general cubic term would also have some terms $?X^3 + ?X^2Y + ?XY^2 + ?Y^3$, but the form is supposed to vanish identically when we plug in $Z = T = 0$. We see that there is a 16-dimensional vector space of such equations. Modding out by scalars, we get $\mathbb{P}^{15}$.

So we see the second term is proper (i.e. compact) and irreducible of dimension 19. It maps to the first term, which also is compact of dimension 19.

We want this map to be surjective, and if it were, we would be done (as then every cubic surface would contain a line). It suffices to show that the image is dense in $\mathbb{P}^{19}$ (by properness). But we know this: the image includes the locus of smooth surfaces that *are* blow ups of $\mathbb{P}^2$ at 6 points, which we know is dense! $\square$

Fix now a line l in S. Suppose P is a plane containing l. What could $P \cap l$ look like? Answer: l union a conic, by Bezout. What if we see l union a line? Then there has to be a third line in P, possibly passing through the intersection of the first two (draw it).

Side remark. This is a tritangent to the cubic surface. (Show them the napkins.)

Side remark. Any other line in S has to meet one of these 3. Reason: it meets P, and that point is in $P \cap S = l \cup l' \cup l''$.

Remark. If three lines meet at a point, they must be coplanar. Otherwise, at the point of meeting, we'll have three tangent vectors to S that aren't linearly independent!

Lemma. No two of l, l', l'' can coincide.

Proof. Suppose $l = l'' \neq l'$. (The case $l = l' = l''$ is left as an **exercise**.) Then on S, we have $2l + l' = H$ (where H is the hyperplane class). We know that $l \cdot l' = 1$ and $l \cdot H = 1$. Hence $l \cdot l = 0$, contradicting the fact that $l^2 = -1$. □

Proposition. Given any line l in S, there are exactly 10 other lines in S meeting l (distinct from l). These fall into 5 disjoint pairs of concurrent lines.

Key Corollary. Any smooth cubic contains two skew lines.

(Warning: our argument only works in characteristic distinct from 2. But the result is still true!)

Proof. Let's fix coordinates, so that l is given by $Z = T = 0$.

Consider the pencil of planes $P_\lambda = Z + \lambda T$ ($\lambda \in \mathbb{P}^1$) containing l. Then $S \cap P_\lambda$ is l union a conic C_λ.

By the previous lemma, C_λ can't be a double line, nor can it contain l. So it will suffice to show that C_λ is singular for 5 values of λ (as a singular conic is a union of 2 lines).

As we had earlier, S is given by an equation of the form

$$A_1(Z,T)X^2 + 2B_1(Z,T)XY + C_1(Z,T)Y^2 + 2D_2(Z,T)X + 2E_2(Z,T)Y + F_3(Z,T) = 0,$$

Setting $Z = \lambda T$ and dividing by T gives the equation for C_λ.

$$A_1(\lambda,1)X^2 + 2B_1(\lambda,1)XY + C_1(\lambda,1)Y^2 + 2D_2(\lambda,1)TX + 2E_2(\lambda,1)TY + F_3(\lambda,1)T^2 = 0,$$

Thus C_λ is singular just when the determinant

$$\Delta(Z,T) = \det \begin{pmatrix} A_1(\lambda,1) & B_1(\lambda,1) & D_2(\lambda,1)T \\ B_1(\lambda,1) & C_1(\lambda,1) & E_2(\lambda,1)T \\ D_2(\lambda,1)T & E_2(\lambda,1)T & F_3(\lambda,1)T^2 \end{pmatrix} = 0.$$

This is degree 5 in λ, and so we have 5 roots. We are now fervently hoping that there are no double roots; then we'll be done.

Suppose we have a root. This corresponds to one of our two pictures (the first in which l, l', l'' formed a triangle; the second in which they formed an asterisk).

I'll do the first case; the second case is an **exercise** as it is similar.

First case, $ll'l''$ is a triangle. By renaming Z and T, we can take $\lambda = 0$. By renaming X and Y, we can take the singular conic to be $XY = 0$. Hence every entry in the determinant is divisible by λ except for $B_1(\lambda, 1)$. Thus when we expand the determinant out, and discard the terms divisible by λ^3, we have only one term left: $B_1(\lambda, 1)^2 F_3(\lambda, 1)$. We know that B_1 is *not* divisible by λ, and F_3 *is*. We want to check that F_3 is not divisible by λ^2; then Δ is divisible by λ and not λ^2, and hence λ is a single root.

If F were divisible by λ^2, then S has equation

$$A_1(Z,T)X^2 + 2B_1(Z,T)XY + C_1(Z,T)Y^2 + 2ZD_1'(Z,T)X + 2ZE_1'(Z,T)Y + Z^2F_1'(Z,T) = 0.$$

I claim that this is singular at $X = Y = Z = 0$: Substitute $T = 1$, and see that there no linear term.

Second case, $ll'l''$ is an asterisk. As I mentioned earlier, this is similar. To get you started: as in the previous case, we can assume $\lambda = 0$. By renaming X and Y, we can take the singular conic to be $X^2 - T^2 = 0$. Go from there. □

Theorem. S is the blow-up of $\mathbb{P}^2$ at 6 points.

We'll see instead that S is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 5 points. We'd seen earlier that $\mathbb{P}^2$ blown up at 2 points is $\mathbb{P}^1$ blown up at one point, so we'll be done.

Proof. Let l and l' be disjoint lines. We define rational maps $\phi : l \times l' \dashrightarrow S$ and $\psi : S \dashrightarrow l \times L'$ as follows: if (p, p') is a generic point of $l \times l'$, the line $\langle p, p' \rangle$ meets S in a third point p''. Define $\phi(p, p') = p''$. For $s \in S - l - l'$, set $p = l \cap \langle s, l' \rangle$, $p' = l' \cap \langle s, l \rangle$ and put $\phi(s) = (p, p')$. It is clear that ϕ and ψ are inverses. Moreover, ϕ is a morphism: we can define at a points of l (or l) by replacing the plane $\langle s, l \rangle$ by the tangent plane to S at s (checking that this gives a morphism, explain). Thus ψ is a birational morphism, and is a composite of blow-ups. Which curves are blown-down? Those lines meeting both l and l'.

Explain why there are precisely 5 of them, one of each the pairs described above. □

1.1. **An alternate approach.** Another way to see 27 lines, and to show the existence of a line in every smooth surface. Consider $\mathbb{G}(1, 3)$. On this 4-fold, there is a rank 4 vector bundle V, corresponding to cubic forms on the line. Given a smooth cubic surface, we have a cubic form $f(x_0, x_1, x_2, x_3)$ which restricts to a cubic form on each line. Thus we have an induced section of V. If this section is 0 at a point of $\mathbb{G}(1, 3)$, it means that the corresponding line is contained in the cubic surface!

By the theory of Chern classes, if the section has no zeros, then $c_4(\mathbb{G}(1, 3)) = 0$. But in fact one can compute that $c_4(V) = 27[pt]$ (omitted).

This can be used to give an argument that there are precisely 27 lines on every cubic surface. If the section has a finite number of zeros on $\mathbb{G}(1, 3)$, then the number of zeros, counted with multiplicities, is the degree of $c_4(V)$. There are a finite number of zeros because if there were an infinite number, we would have a line that could move on the surface. But any line has self-intersection -1 and hence doesn't move. Furthermore, a careful local calculation shows that this line counts with multiplicity 1.

This requires more technique, so I've omitted the details. This is perhaps a faster way to show that there are 27 lines on every cubic surface, but it doesn't show you that the surface is $\mathbb{P}^2$ blown up at 6 points.

2. Castelnuovo's Theorem

We saw how tricky it was to show that a surface is rational. On Friday, we will prove:

Theorem: Castelnuovo's Rationality Criterion. Let S be a surface with $q = P_2 = 0$. Then S is rational.