COMPLEX ALGEBRAIC SURFACES CLASS 13

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1. Fun with rational surfaces 1

Last day we began:

1. FUN WITH RATIONAL SURFACES

I'll start again, but skip over things we did last time.

The surface $\mathbb{F}_0$. This is $\mathbb{P}^1 \times \mathbb{P}^1$. Its intersection theory is $\mathbb{Z}[h, f]/h^2 = f^2 = 0, hf = 1$. h and f are the classes of fibers of the two projections to $\mathbb{P}^1$. These are traditionally called *rulings*. Using the divisor $h + f$, we can embed $\mathbb{F}_0$ as a smooth quadric in $\mathbb{P}^3$. More precisely than last day: Let $([x; y], [u; v])$ be co-ordinates to $\mathbb{P}^1 \times \mathbb{P}^1$. Then xu, xv, yu, yv are sections of $\mathcal{O}(h + f)$, that separate points and tangent vectors, and hence give a closed immersion into $\mathbb{P}^3$. If the base field is algebraically closed, all smooth quadrics are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Now, what do the rulings look like in projective space? What's their degree? Answer: $h \cdot (h + f) = 1$. They are lines! (Draw pictures of a hyperboloid with one sheet, and show them the lines.) Question: where are the lines in an ellipsoid? Answer: they are complex, so don't be misled by the real picture.

Many of the other surfaces corresponds to blow-ups of $\mathbb{P}^2$ at a certain number of points. Before discussing them, here is a useful proposition that I mentioned last day.

Useful proposition. Consider the blow-up of $\mathbb{P}^2$ at n general points, giving exceptional divisors $E_1, \ldots, E_n$. Then the intersection ring on $\mathbb{P}^2$ is given by

$$\mathbb{Z}[H, E_1, \ldots, E_n]/H^2 = 1, HE_i = 0, E_i \cdot E_j = 0, E_i^2 = -1.$$

We can understand divisors and sections of divisors in terms of divisors on $\mathbb{P}^2$ with certain multiplicities at the E_j. More precisely: the vector space of sections of $aH - b_1E_1 - \cdots - b_nE_n$ is naturally isomorphic to the vector space of degree a polynomials in $\mathbb{P}^2$ vanishing with multiplicity at least b_i on E_i.

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Sketch of proof. Any divisor in $\mathcal{O}(aH - b_1E_1 - \cdots - b_nE_n)$ pushes forward to some divisor D in class aH on $\mathbb{P}^2$. The strict transform of that divisor (i.e., somewhat hideously: the closure of its preimage where the blow-up is an isomorphism) is in class $aH - (\operatorname{mult}_{p_1} D)E_1 - \cdots - (\operatorname{mult}_{p_n} D)E_n$. So the actually original divisor must be this, plus some more E_i's, from which $b_i \leq \operatorname{mult}_{p_1} D$.

The vector space structure is the same, as both are subvector spaces of the sections over $\mathbb{P}^2 - p_1 - \cdots - p_n$. $\square$

The surface $\mathbb{F}_1$. As observed before, $\mathbb{F}_1$ is (isomorphic to) the blow-up of $\mathbb{P}^2$ at a point.

Consider the divisor class $2H - E_1$. This corresponds to conics in $\mathbb{P}^2$ through p_1, which gives a five-dimensional vector space. It separates points and tangent vectors (somewhat obviously away from E_1, and not so obviously along E_1). Thus we get an immersion of $\mathbb{F}_1$ into $\mathbb{P}^4$. Its degree is $(2H - E) \cdot (2H - E) = 3$. We get a cubic surface in $\mathbb{P}^4$.

Interpretation as projection from Veronese surface. Recall that we had an embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ via all conics. We can interpret $\mathbb{F}_1$ and its cubic embedding in $\mathbb{P}^4$ as a projection as follows. Suppose $\mathbb{F}_1 = \operatorname{Bl}_{[0;0;1]} \mathbb{P}^2$.

$$\begin{array}{ccc}
\mathbb{P}^2 & \stackrel{[x_0^2,x_0x_1;x_1^2;x_0x_2;x_1x_2;x_2^2]}{\hookrightarrow} & \mathbb{P}^5 \\
\uparrow & \nearrow & \downarrow \text{ should be dashed} \\
\mathbb{F}_1 & \stackrel{[x_0^2,x_0x_1;x_1^2;x_0x_2;x_1x_2]}{\hookrightarrow} & \mathbb{P}^4.
\end{array}$$

Project the quartic Veronese surface in $\mathbb{P}^5$ down into $\mathbb{P}^4$ from $[0;0;0;0;0;1]$. This is well defined except at that one point of the Veronese surface. We have a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^4$. By the elimination of indeterminacy theorem, we can resolve the map after some blow-ups of $\mathbb{P}^2$. In fact, we need only one: $\mathbb{F}_1 \to \mathbb{P}^2$.

Blow up $\mathbb{P}^2$ at two points. Now blow up $\mathbb{P}^2$ at two points. Where are the (-1)-curves? There are obviously two: our two exceptional divisors. But there is one more: the proper transform of the line joining the two points. It has self-intersection number $(H - E_1 - E_2)^2 = -1$.

When you blow down that "bonus" rational curve, what do you get? In fact: $\mathbb{P}^1 \times \mathbb{P}^1$. Last time, Eric saw this by interpreting what we just did as an elementary transformation of $\mathbb{F}_1$.

Here's another way of seeing it. The divisor $2H - E_1 - E_1$ corresponds to conic through our points, and is very ample. It gives an immersion into $\mathbb{P}^3$, and it is degree 2. Get smooth quadric surface. But we know that all smooth quadrics are $\mathbb{P}^1 \times \mathbb{P}^1$.

What are the maps to the two $\mathbb{P}^1$'s? Answer: projection from each of two points. You can see why the proper transform of the line gets sent to a point under these two projections.

Blow up $\mathbb{P}^2$ at three points, no two on a line. Where are the (-1)-curves? Answer: the 3 exceptional divisors E_1, E_2, E_3, but also the (proper transforms of) the lines through pairs of those points $H - E_i - E_j$.

We can make an diagram showing how these six curves intersect, with a vertex for each curve, and an edge for each intersection. In this case, we get a hexagon.

If you blow down those other 3 curves? Answer: $\mathbb{P}^2$. The rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is ancient, and is called a Cremona transformation.

Description 1: conics through these 3 points, i.e. $2H - E_1 - E_2 - E_3$. The space of all conics is a six-dimensional vector space, and if we require the conics to vanish at the three points, we knock the vector space down to dimension 3. So this gives a map to $\mathbb{P}^2$. It blows down the line $H - E_1 - E_2$.

Description 2: Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $[x; y; z] \mapsto [1/x; 1/y; 1/z]$. By the elimination of indeterminacy theorem, we can resolve this map after some blow-ups. In fact, we need 3.

For fun, let's look more closely. If $x = 0$ and $y, z \neq 0$, then we map to $[1; 0; 0]$; that's a line being blown down. If $x = 0$ and $y = 0$ and $z \neq 0$, it isn't so clear. In fact, if $[x; y; z] = [at, bt, 1]$ where a and b aren't both 0, then $[x; y; z] \mapsto [1/a; 1/b; t]$. Then let t go to 0, and we see that the limit is $[1/a; 1/b; 0]$. Thus the limit depends on the path of approach, and this is a point where the rational map is undefined, and will have to be blown up. That's p_i. **And so on...**

Aside: What happens if you blow up at 3 points on a line? Get a (-2)-curve.

Blow up $\mathbb{P}^2$ at four points. Blow up at 4 general points. Get $10 = 4 + \binom{4}{2}$ (-1)-curves.

Make graph. This is a highly symmetric graph, and is called the *Peterson graph*. Its automorphism group is S_5; both Tyler and Diane gave me arguments for this within minutes of the end of class.

Blow up $\mathbb{P}^2$ at five points. Blow up at 5 general points. Get $5 + \binom{5}{2} = 15$, plus one more: conic. Cubics through these points. Get quartic surface in $\mathbb{P}^4$. (It is the complete intersection of two complete quadrics.

The (-1)-curves turn into lines: $E_1 \cdot (3H - \sum E_i) = 1$. $(H - E_1 - E_2) \cdot (3H - \sum E_i) = 1$. $(2H - \sum E_i) \cdot (3H - \sum E_i) = 1$.

We could have saved ourselves some effort by instead noting that $K_S = -3H + \sum E_i$. The divisor we are using to embed is $-K_S$. If E is a (-1)-curve, then we know that $E^2 = -1$, and by the genus formula $(K_S + E) \cdot E = -2$, from which $(-K_S \cdot E) = 1$. Note that we can reverse this: if S is embedded by the anticanonical divisor, then lines correspond to (-1)-curves. **Exercise.**

Blow up $\mathbb{P}^2$ at six points. Now this is serious. Blow up at 6 points: Get $6 + \binom{6}{2} + \binom{6}{5} = 6 + 15 + 6 = 27$.

Get smooth cubic surface in $\mathbb{P}^3$! With 27 lines! We'll prove (next day) (i) there are no more lines on this surface, (ii) that almost all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points and hence have 27 lines, (iii) all smooth cubic surfaces have 27 lines, and (iv) all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points.

The automorphism group of the graph of exceptional curves is $W(E_6)$, the Weyl group of E_6.

Blow up $\mathbb{P}^2$ at seven, eight, nine points.

Get 56, [I can't remember the number, maybe 148], ∞ lines. More interesting geometry here too. For example, blow up at 7 points. Get map to $\mathbb{P}^2$. It is a double cover. (-1)-curves map to lines. Fact: the branch locus is a quartic plane curve. (-1)-curves map 2-to-1 to the 28 bitangents of a smooth quartic plane curve.

The automorphism group of the graph of exceptional curves is $W(E_n)$, for $n = 7, 8, 9$.