COMPLEX ALGEBRAIC SURFACES CLASS 12

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CONTENTS

Last day:

Lemma: All rank 2 locally free sheaves are filtered nicely by invertible sheaves. Suppose E is a rank 2 locally free sheaf on a curve C.

(i) There exists an exact sequence $0 \to L \to E \to M \to 0$ with $L, M \in \operatorname{Pic} C$. Terminology: E is an *extension* of M by L.
(ii) If $h^0(E) \geq 1$, we can take $L = \mathcal{O}_C(D)$, with D the divisor of zeros of a section of E. (Hence D is effective, i.e. $D \geq 0$.)
(iii) If $h^0(E) \geq 2$ and $\deg E > 0$, we can assume $D > 0$.

(i) is the most important one.

We showed that extensions $0 \to L \to E \to M \to 0$ are classified by $H^1(C, L \otimes M^*)$. The element 0 corresponded to a splitting. If one element is a non-zero multiple of the other, they correspond to the same E, although different extensions.

As an application, we proved: **Proposition.** Every rank 2 locally free sheaf on $\mathbb{P}^1$ is a direct sum of invertible sheaves.

I can't remember if I stated the implication:

Every geometrically ruled surface over $\mathbb{P}^1$ is isomorphic to a Hirzebruch surface

$$\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

for $n \geq 0$.

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(We don't yet know that they are all different yet, but we will soon.)

We then began to study :

1. Geometric facts about a geometrically ruled surface $\pi : S = \mathbb{P}_C E \to C$ from geometric facts about C

In particular, we found the intersection theory of S in terms of $\operatorname{Pic} C$. One of the players was the class $\mathcal{O}(1)$ of the projective bundle. Let me repeat the definition of this.

There is a "tautological" subline bundle of $\pi^* E$; this is defined to be $\mathcal{O}_P(-1)$ (and $\mathcal{O}_P(n)$ is defined to be the appropriate multiple = tensor power of this).

You can check:

Exercise.

(a) This agrees with the definition of $\mathcal{O}(1)$ on a projective space (in the case where E is a point).
(b) In the general case, the restriction of $\mathcal{O}_P(1)$ to a fiber of π is $\mathcal{O}(1)$ on the fiber.
(c) In the case where $\dim C = 1$ and $\dim P = 2$, $\mathcal{O}(1) \cdot F = 1$ for any fiber F. (This generalizes to projective bundles of arbitrary dimension once we have intersection theory.)
(d) Hence if L is any invertible sheaf on C, then $\pi^* L \cdot \mathcal{O}(1) = \deg L$.

Once you check (a), the rest follow quickly in order.

With this definition, we have:

Proposition. Suppose $\pi : S \to C$ is a geometrically ruled surface, corresponding to rank 2 locally free sheaf E. Let $h = \mathcal{O}_S(1) \in \operatorname{Pic} S$ or $H^2(S, \mathbb{Z})$ Then:

(i) $\operatorname{Pic} S = \pi^* \operatorname{Pic} C \oplus \mathbb{Z} h$,
(ii) $H^2(S, \mathbb{Z}) = \mathbb{Z} h + \mathbb{Z} f$, where f is the class of a fiber,
(iii) $h^2 = \deg E$,
(iv) $[K] = -2h + (\deg E + 2g(C) - 2) f$ in $H^2(S, \mathbb{Z})$.

I gave a proof of this, but my proof of (iv) needs to be edited. The corrected version is in the Class 11 notes. (Remark: The reason for the discrepancy is because there are two possible definitions of the projectivization of a vector space. The traditional one is that of one-dimensional subspaces of a vector space. That's the one that most of the world uses. An alternate one is that of one-dimensional quotients of a vector space. That's the one that Grothendieck used, because it makes certain statements cleaner, and as a result much but not of the algebraic geometry community uses this definition. So be warned.)

2. THE HODGE DIAMOND OF A RULED SURFACE

Recall that the outer entries of a Hodge diamond are birational invariants. I should have proved this earlier, and will now. (Move to earlier notes at some point.)

Theorem. $h^0(S, \Omega_S) = h^{1,0}$ is a birational invariant. (Similarly for $h^{2,0}$, and also so for pluricanonical forms $P_n = H^0(S, \mathcal{K}_S^{\otimes n})$.) (This works for smooth projective varieties of any dimension.)

Proof. Suppose $\phi : S' \dashrightarrow S$ is a birational map, which is a morphism from $S' - F \to S$, where F is a finite set. Thus we have a map $H^0(S, \Omega_S) \to H^0(S' - F, \Omega_{S'-F})$. In fact, this extends: the poles of a differential form are pure codimension 1. Thus we have $H^0(S, \Omega_S) \to H^0(S', \Omega_{S'})$ which takes a preserves the restriction of the form to their common open set. The same argument works for the rational map $S \to S'$, so we get an isomorphism. □

Hence the numerical invariants of a ruled surface $\pi : S \to C$ are as follows. The outer entries of the Hodge diamond are the same as for $C \times \mathbb{P}^1$, which can be checked directly to be

$$\begin{array}{ccccc} & & h^{0,0} = 1 & & \\ & h^{1,0} = g & & h^{0,1} = g & \\ h^{2,0} = 0 & & h^{1,1} & & h^{0,2} = 0 \\ & h^{2,1} = g & & h^{1,2} = g & \\ & & h^{2,2} = 1 & & \end{array}$$

The central number is the rank of the Picard group (by the Lefschetz (1,1)-theorem), or over $\mathbb{C}$ can be found using the Euler characteristic.

Also, $P_n = H^0(S, \mathcal{K}_S^{\otimes n}) = 0$. Important fact: this characterizes ruled surfaces. (I doubt we'll get to this fact in this course, but we might.)

Second important fact: if $q = P_2 = 0$, then S is rational. (Castelnuovo's Rationality Criterion, to be discussed in a few classes.)

3. THE SURFACES $\mathbb{F}_n$

We already know that $\operatorname{Pic} \mathbb{F}_n = \mathbb{Z}h \oplus \mathbb{Z}f$ ($n \geq 0$) with $f^2 = 0$, $f \cdot h = 1$, and $h^2 = n$. All geometrically ruled surfaces over $\mathbb{P}^1$ are of this sort. But are they all different? Yes!:

Proposition. If $n > 0$, there is a unique irreducible curve E on $\mathbb{F}_n$ with negative self-intersection. If e is its class in $\operatorname{Pic} \mathbb{F}_n$, then $e = h - nf$.

(This curve is often called E.)

Corollary. $e^2 = -n$. Note that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ has no curves of negative self-intersection, as they all "move" (under the automorphisms of both $\mathbb{P}^1$'s). Hence $\mathbb{F}_n$ and $\mathbb{F}_m$ are not isomorphic unless $n = m$. Also, all $\mathbb{F}_n$ are minimal except for $\mathbb{F}_1$, which has a (-1)-curve. In fact, $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at a point. Reason: the blow-up of $\mathbb{P}^2$ at a point is a rational geometrically ruled surface with a (-1)-curve, so it must be $\mathbb{F}_1$!

Proof of the proposition. First, I'll produce the curve E of negative self intersection, which is a section. Consider the section E of $\pi : \mathbb{F}_n \to \mathbb{P}^1$ corresponding to the subline bundle $\mathcal{O}_{\mathbb{P}^1}$. Let e be its class of this curve E in $\operatorname{Pic} \mathbb{F}_n$. $e = h + rf$ for some r. Since $\mathcal{O}_{\mathbb{F}_n}(1)|_E = \mathcal{O}_{\mathbb{P}^1}$, we know that $h \cdot e = \deg \mathcal{O}_{\mathbb{F}_n}(1)|_E = 0$, from which $r = -n$, and then $e^2 = (h - nf)^2 = -n$.

Next, I'll show that this is the only curve of negative self-intersection. If C is irreducible on $\mathbb{F}_n$. $[C] = ah + bf$. $C \cdot f = 0$ implies $a \geq 0$. $C \cdot E \geq 0$ implies $(bf \cdot (h - nf)) = b \geq 0$. Finally, $C \cdot C = (ah + bf) = a^2 n + 2ab \geq 0$. □

3.1. **Getting from one $\mathbb{F}_n$ to another by elementary transformations.** Consider $\mathbb{F}_n$. Blow up a point on E, and blow down the proper transform of the fiber. We again have a rational ruled surface. Show that you have $\mathbb{F}_{n+1}$.

Instead, blow up a point not on E, and blow down the proper transform of the fiber. Show that you have $\mathbb{F}_{n-1}$. What goes wrong when $n = 0$?

4. Fun with rational surfaces (beginning)

The surface F_0. This is $\mathbb{P}^1 \times \mathbb{P}^1$. By the above determination of the intersection theory of $\mathbb{F}_n$, the intersection theory of $\mathbb{F}_0$ is $\mathbb{Z}[h, f]/h^2 = f^2 = 0, hf = 1$. h and f are the classes of fibers of the two projections to $\mathbb{P}^1$. Let $([x; y], [u; v])$ be co-ordinates to $\mathbb{P}^1 \times \mathbb{P}^1$. Then xu, xv, yu, yv are sections of $\mathcal{O}(h + f)$, that separate points and tangent vectors, and hence give a closed immersion into $\mathbb{P}^3$. The degree of the immersed surface is $(h + f)^2 = h^2 + 2hf + f^2 = 0 + 2 + 0 = 2$, so we have a smooth quadric in $\mathbb{P}^3$. Conversely, any two smooth quadrics over an algebraically closed field are isomorphic. The reason is linear algebra: quadratic forms in 4 variables are classified, and are (up to change of basis):

$$0, \quad x_0^2, \quad x_0^2 + x_1^2, \quad x_0^2 + x_1^2 + x_2^2, \quad x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

Only the last one is smooth. (The rest are: all of projective space, a double plane, the union of two planes, and a quadric cone.)

Thus all smooth quadrics in $\mathbb{P}^3$ (over an algebraically closed field) are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Many of the other surfaces corresponds to blow-ups of $\mathbb{P}^2$ at a certain number of points. Before discussing them, here is a

Useful proposition. Consider the blow-up of $\mathbb{P}^2$ at n general points, giving exceptional divisors $E_1, \ldots, E_n$. Then the intersection ring on $\mathbb{P}^2$ is given by

$$\mathbb{Z}[H, E_1, \ldots, E_n]/H^2 = 1, HE_i = 0, E_i \cdot E_j = 0, E_i^2 = -1.$$

We can understand divisors and sections of divisors in terms of divisors on $\mathbb{P}^2$ with certain multiplicities at the E_j. More precisely: the vector space of sections of $aH - b_1E_1 - \cdots - b_nE_n$ is naturally isomorphic to the vector space of degree a polynomials in $\mathbb{P}^2$ vanishing with multiplicity at least b_i on E_i.

I'll prove this next class.

The surface $\mathbb{F}_1$. As observed before, $\mathbb{F}_1$ is (isomorphic to) the blow-up of $\mathbb{P}^2$ at a point.

Consider the divisor class $2H - E_1$. This corresponds to conics in $\mathbb{P}^2$ through p_1, which gives a five-dimensional vector space. It separates points and tangent vectors (somewhat obviously away from E_1, and not so obviously along E_1). Thus we get an immersion of $\mathbb{F}_1$ into $\mathbb{P}^4$. Its degree is $(2H - E) \cdot (2H - E) = 3$. We get a cubic surface in $\mathbb{P}^4$.

Blow up $\mathbb{P}^2$ at two points. Now blow up $\mathbb{P}^2$ at two points. Where are the (-1)-curves? There are obviously two: our two exceptional divisors. But there is one more: the proper transform of the line joining the two points. It has self-intersection number $(H - E_1 - E_2)^2 = -1$.

When you blow down that "bonus" rational curve, what do you get? In fact: $\mathbb{P}^1 \times \mathbb{P}^1$. Eric saw this by interpreting what we just did as an elementary transformation of $\mathbb{F}_1$.

Here's another way of seeing it. Conic through 2 points ($2H - E_1 - E_2$) gives a map to $\mathbb{P}^3$, and it is degree 2. Get smooth quadric surface.

What are the maps to the two $\mathbb{P}^1$'s? Answer: projection from each of two points. You can see why the proper transform of the line gets sent to a point under these two projections.