COMPLEX ALGEBRAIC SURFACES CLASS 11

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1. GEOMETRICALLY RULED SURFACES AND PROJECTIVIZATIONS OF RANK 2 LOCALLY FREE SHEAVES

Last time, we began analyzing geometrically ruled surfaces by studying projectivizations of rank 2 locally free sheaves.

We proved:

Proposition. Every geometrically ruled surface over C is C-isomorphic to $\mathbb{P}_C(E)$ for some rank 2 locally free sheaf (vector bundle) over C. The bundles $\mathbb{P}_C(E)$ and $\mathbb{P}_C(E')$ are isomorphic (over C) iff there is an invertible sheaf (line bundle) L on C such that $E' \cong E \otimes L$.

I then stated the first part of the following lemma.

Lemma: All rank 2 locally free sheaves are filtered nicely by invertible sheaves. Suppose E is a rank 2 locally free sheaf on a curve C.

(i) There exists an exact sequence $0 \to L \to E \to M \to 0$ with $L, M \in \operatorname{Pic} C$. Terminology: E is an *extension* of M by L.
(ii) If $h^0(E) \geq 1$, we can take $L = \mathcal{O}_C(D)$, with D the divisor of zeros of a section of E. (Hence D is effective, i.e. $D \geq 0$.)
(iii) If $h^0(E) \geq 2$ and $\deg E > 0$, we can assume $D > 0$.

(i) is the most important one.

I mentioned the application of *Riemann-Roch for rank 2 vector bundles on a curve*:

$$\chi(E) = \deg E + 2(1-g).$$

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Second: the filtrations correspond to sections of the projective bundle. Hence all projective bundles over curves have sections.

Proof. (i) We can twist E by some invertible sheaf N so that it has a non-zero section s. Here's how: take a rational section of E. It has some poles of various orders at various points. Twist by the invertible sheaf allowing these poles.

The section s gives $\mathcal{O}_C \to E \otimes N$.

Here's an argument which *doesn't* work. (Find the mistake.) The cokernel is locally free, call it M', and the kernel of $E \otimes N \to M'$, which is a subsheaf of $\mathcal{O}_C$, is necessarily $\mathcal{O}_C(D)$ for D the zero-divisor of the section s. Hence we have

$$0 \to \mathcal{O}_C(D) \to E \otimes N \to M' \to 0.$$

Twist this by N^*, and we get (i).

In fact, the cokernel needn't be an invertible sheaf! (This is the only flaw.) In general, the cokernel of a morphism of locally free sheaves needn't be locally free! (This is *different* behavior than for vector bundles, and is one good reason to keep the two concepts separate in your mind.) You've seen an example before: If X is a variety and D a divisor, consider the morphism of invertible sheaves $\mathcal{O}_X(-D) \to \mathcal{O}_X$. then we have $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$. (If you want to see a morphism from a rank 1 locally free to a rank 2 locally free, tweak this to get $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \oplus \mathcal{O}_X \to \mathcal{O}_D \oplus \mathcal{O}_X \to 0$.) Hence this morphism *doesn't correspond to* a morphism of vector bundles. How to patch? We'll use the useful fact is that *any subsheaf of a locally free sheaf on a smooth curve is also locally free*. (The proof is omitted. The key is to show that torsion-free sheaves on a smooth curve are locally free, which is an algebraic fact.) *Dualize* the problematic morphism $\mathcal{O}_X \to E \otimes N$ to get $E^* \otimes N^* \to \mathcal{O}_X$. Take the *image* of $E^* \otimes N^*$ in $\mathcal{O}_X$. By the useful fact, the image is also an invertible sheaf $\mathcal{O}_X(-D)$ (check first that the image isn't 0). The kernel of this morphism is a subsheaf of a locally free sheaf, and is thus also locally free, in fact of rank 1. So we get

$$0 \to M' \to E^* \otimes N^* \to \mathcal{O}(D) \to 0.$$

Dualizing gives us $0 \to \mathcal{O}(D) \stackrel{(*)}{\to} E \otimes N \to (M')^* \to 0$.

Useful facts (proofs omitted): (i) the morphism (*) corresponds to a section of $E \otimes N$, with poles along D. (ii) to translate to vector bundles, a morphism of vector bundles on a smooth variety corresponds to a morphism of locally free sheaves *whose cokernel is locally free* (the kernel automatically is). (iii) related fact: sections of the projective bundle correspond to filtrations.

For (ii), note that if $h^0(E) > 0$, then E already has a non-zero section, and we can omit the twist by N. Then that last exact sequence is the one we desire.

For (iii), it suffices to show that there is a section of E that vanishes at some point. Let s and t be two linearly independent sections. As $\deg E > 0$, the section $s \wedge t$ of $\wedge^2 E$ must vanish at some point $p \in C$. That means at that point there are α and β (not both 0) such that $\alpha s(p) + \beta t(p) = 0$. So the section $\alpha s + \beta t$ vanishes at p, proving (iii). $\square$

Hence we want to classify extensions of invertible sheaves. Up to twists by invertible sheaves, these correspond to projective bundles with section.

Given an extension $0 \to L \to E \to M \to 0$, we get a class of $H^1(C, L \otimes M^*)$ as follows. (This has nothing to do with C being a curve, or with L being invertible.) Twist by M^*, take the long exact sequence, and look at the image of $1 \in H^0(C, \mathcal{O}_C)$. Extensions are classified precisely by this cohomology group. In other words, two extensions are isomorphic

$$\begin{array}{ccccccccc} 0 \to & L & \to & E & \to & M & \to 0 \\ & || & & \downarrow\sim & & || & \\ 0 \to & L & \to & E' & \to & M & \to 0 \end{array}$$

if they induce the same element of $H^1(C, L \otimes M^*)$. One direction is now clear.

Exercise: Check the other direction. (Hint, useful in other circumstances: Given an element of $H^1(C, L \otimes M^*)$, say in Cech cohomology, recover the extension.) Check that the $0 \in H^1(C, L \otimes M^*)$ corresponds to $L \oplus M$. Check that if $a = kb$ where $a, b \in H^1(C, L \otimes M^*)$ and $k \neq 0$, then $E_a \cong E_b$.

Proposition. Every rank 2 locally free sheaf on $\mathbb{P}^1$ decomposes into the sum of two invertible sheaves. Hence every geometrically ruled surface over $\mathbb{P}^1$ is isomorphic to a Hirzebruch surface

$$\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

for $n \geq 0$.

Hence every geometrically ruled surface is one of the $\mathbb{F}_n$'s described earlier. (We don't yet know that they are all different yet.)

Proof. We can twist by a line bundle so as to assume that $\deg E = 0$ or 1. By Riemann-Roch, $h^0(E) \geq 1$, so there is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(k) \to E \to \mathcal{O}_{\mathbb{P}^1}(d-k) \to 0$$

with $k \geq 0$. But these extensions are classified by $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2k-d)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2-2k+d))^* = \{0\}$, so this must be a direct sum. $\square$

Exercise: Grothendieck's Theorem. Show that every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles.

Similarly, you can prove:

Proposition.

(a) Every rank 2 vector bundle on an elliptic curve is either decomposable, or isomorphic to $E \otimes L$, where $L \in \operatorname{Pic} C$ and E is either (i) the unique non-trivial extension of $\mathcal{O}_C$ by $\mathcal{O}_C$, or (ii) the non-trivial extension of $\mathcal{O}_C(p)$ by $\mathcal{O}_C$ for some p. (**Exercise**.)

(b) For every curve C of genus g, there exist families of rank 2 vector bundles parametrized by some variety S (possibly singular, non-compact) of dimension at least $2g-3$. (See Beauville.)

2. Geometric facts about geometrically ruled surfaces over C, from geometric facts about C

(This should be moved to early in the course notes.) First, we'll need the definition of $\mathcal{O}(1)$ for a projective bundle $\pi : \mathbb{P}E \to B$. There is a "tautological" subline bundle of $\pi^* E$; this is defined to be $\mathcal{O}_P(-1)$ (and $\mathcal{O}_P(n)$ is defined to be the appropriate multiple = tensor power of this).

You can check:

Exercise.

(a) This agrees with the definition of $\mathcal{O}(1)$ on a projective space (in the case where E is a point).
(b) In the general case, the restriction of $\mathcal{O}_P(1)$ to a fiber of π is $\mathcal{O}(1)$ on the fiber. (This is basically immediate from (a).)
(c) In the case where $\dim B = 1$ and $\dim P = 2$, $\mathcal{O}(1) \cdot F = 1$ for any fiber F. (This is basically immediate from (b); and it generalizes to projective bundles of arbitrary dimension once we have intersection theory.)

Proposition. Suppose $\pi : S \to C$ is a geometrically ruled surface, corresponding to rank 2 locally free sheaf E. Let $h = \mathcal{O}_S(1) \in \operatorname{Pic} S$ or $H^2(S, \mathbb{Z})$ Then:

(i) $\operatorname{Pic} S = \pi^* \operatorname{Pic} C \oplus \mathbb{Z}h$,
(ii) $H^2(S, \mathbb{Z}) = \mathbb{Z}h + \mathbb{Z}f$, where f is the class of a fiber,
(iii) $h^2 = \deg E$,
(iv) $[K] = -2h + (\deg E + 2g(C) - 2)f$ in $H^2(S, \mathbb{Z})$.

Remarks.

- This tells us about the intersection theory on S.
- (i) and (ii) are true for $\mathbb{P}^1$-bundles over an arbitrary base. There are also analogues of (iii) and (iv) over an arbitrary base.
- Note that $h \cdot f = 1$.
- (ii) follows from (i), as H^2 is a quotient of Pic.
- Assuming (ii) and (iii), proof of (iv) is an **exercise**: $[K] = ah + bf$ in $H^2(S, \mathbb{Z})$. Use the genus formula for a fiber F to get $a = -2$, and the genus formula for a section to get b.

Proof of (i). We get a map from $\pi^* \operatorname{Pic} C \oplus \mathbb{Z}h \to \operatorname{Pic} S$. It is injective: suppose $(\pi^* \mathcal{L}, nh) \mapsto 0$; then by restricting to a fiber F, we get $n = 0$; by restricting to a section, we get $\mathcal{L} = 0$.

Surjectivity: Any element of $\operatorname{Pic} S$ is of the form $D + mh$ where $D \cdot F = 0$. I claim that $D = \pi^* D'$ for some D' on C. Here's why. Consider $D_n := D + nF$.

I claim that $h^0(K - D_n) = 0$ for $n \gg 0$. Reason: take a very ample divisor class $[H]$ on S, so $[H] \cdot F > 0$. Choose n big enough that $[H] \cdot (K - D_n) < 0$. If $h^0(K - D_n) > 0$, then there is a non-zero section. There is a section of $\mathcal{O}(H)$ whose zero set is an effective curve meeting this zero-set. But $H \cdot (K - D_n) < 0$, contradiction.

Now $D_n^2 = D^2$, and $D_n \cdot K = D \cdot K - nF \cdot K = D \cdot K - 2n$, so by Riemann-Roch:

$$h^0(D_n) - h^1(D_n) + h^2(D_n) = \chi(\mathcal{O}_S) + \frac{1}{2}(D_n^2 - D_n \cdot K)$$
$$\Rightarrow \qquad h^0(D_n) \geq 0 + \frac{1}{2}(D^2 - D \cdot K + 2n) > 0.$$

Let $E \in |D_n|$ be the zero-set of a non-zero section. Then $E \cdot F' = 0$ for every fiber F', so E must be a union of fibers with multiplicity, i.e. E is the pullback of some points on C.

Proof of (iii).) Define $c_2(E) := \chi(\mathcal{O}_S) - \chi(E) + \chi(\wedge^2 E)$. Motivation: If $0 \to L \to E \to M \to 0$, we want $c_2(E) = L \cdot M$. Well,

$$L \cdot M = L^* \cdot M^* = \chi(\mathcal{O}_S) - \chi(L) - \chi(M) + \chi(L \otimes M) = \chi(O_S) - \chi(E) + \chi(\wedge^2 E) = c_2(E).$$

Apply this now to $\pi^* E$ on S. There is an exact sequence on C: $0 \to L \to E \to M \to 0$, so $c_2(\pi^* E) = (\pi^* L \cdot \pi^* M) = 0$. **From here on, this argument is "dual" to the one I presented in class, which used a different definition of projective bundle.** Also,

$$0 \to \mathcal{O}(-1) \to \pi^* E \to Q \to 0.$$

Hence $h \cdot [Q] = 0$. Also, taking the "determinant" of the short exact sequence, we get $Q \cong (\pi^* \wedge^2 E) \otimes \mathcal{O}_S(1)$, from which $[Q] = h + \pi^*[\wedge^2(E)]$. Hence

$$0 = h \cdot [Q] = h^2 + h \cdot \pi^*(\wedge^2 E) = h^2 + \deg E,$$

and we're done. $\square$

Remark. Now that we've defined c_2, we can describe Noether's theorem as $\chi(\mathcal{O}_S) = \frac{1}{12}(c_2(T_S) + K_S^2)$.