# COMPLEX ALGEBRAIC SURFACES CLASS 10

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CONTENTS



To avoid constantly rederiving the value of $h^1(\mathcal{O}_{\mathbb{P}^1}(n))$, let me just note for future (and past) reference that it vanishes if $n \geq -1$, by our calculations of $h^0$ and Serre duality or Riemann-Roch.

## 1. Ruled surfaces, continued

Last day, we began talking about ruled surfaces.

A surface is *ruled* if it is birationally equivalent to $C \times \mathbb{P}^1$, where $C$ is a smooth curve.

A surface $S \to C$ is *geometrically ruled* such that the fibers are all isomorphic to $\mathbb{P}^1$. We're in process of proving that this is equivalent to

(a) $S$ is a $\mathbb{P}^1$-bundle over $C$,
(b) $S$ is the projectivization of a rank 2 vector bundle $E$ over $C$.

Clearly (b) implies (a) implies geometrically ruled.

The theorem we're in the process of proving is:

**Noether-Enriques Theorem.** Suppose $\pi : S \to C$ is geometrically ruled. Then $S$ is of type (b) above, i.e. it is the projectivization of some rank 2 invertible sheaf / vector bundle.

We're proving more generally: Suppose $\pi : S \to C$, and $x \in C$ such $\pi$ is smooth over $C$ and $\pi^{-1}(x)$ is isomorphic to $\mathbb{P}^1$. Then there is a Zariski-open subset $U \subset C$ containing $x$

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and a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U \times \mathbb{P}^1 \arrow[dl] \\ & U. & \end{tikzcd}$$

Three-step proof. At this point, we've completed step 2, and we know that there is a divisor $H$ of $S$ such that $H \cdot F = 1$.

Hence the result will follow from:

**Proposition.** Suppose $\pi : X \to Y$ (with no restrictions on dimension), and $x \in Y$, $\pi^{-1}(x)$ is isomorphic to $\mathbb{P}^1$, and there is a divisor $H$ on $X$ meeting $\pi^*(x)$ with multiplicity 1. Then there is a Zariski-open subset $U \subset Y$ containing $x$ and a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U \times \mathbb{P}^1 \arrow[dl] \\ & U. & \end{tikzcd}$$

*Proof.* I'll prove this in the case where $\dim Y = 1$ and $Y$ is smooth, but if you're quite happy with the argument, make the necessary adjustments in your head to make it work. To what extent can you do away with smoothness assumptions?

We now have our line bundle $H$ such that $H \cdot F = 1$, where $F = \pi^{-1}(x)$. Twist

$$0 \to \mathcal{O}_S(-F) \to \mathcal{O}_S \overset{rest.}{\to} \mathcal{O}_F \to 0$$

by $H + rF$ to get

$$0 \to \mathcal{O}_S(H + (r-1)F) \to \mathcal{O}_S(H + rF) \overset{rest.}{\to} \mathcal{O}_F(1) \to 0.$$

Take the long exact sequence:

$$H^0(S, \mathcal{O}_S(H+rF)) \overset{rest.}{\to} H^0(F, \mathcal{O}_F(1)) \to H^1(S, \mathcal{O}_S(H+(r-1)F)) \to H^1(S, \mathcal{O}_S(H+rF)) \to 0.$$

For $r \gg 0$, the dimension of $H^1(S, \mathcal{O}_S(H + rF))$ must stabilize. So restrict to this range, and we get:

$$H^0(S, \mathcal{O}_S(H + rF)) \overset{rest.}{\to} H^0(F, \mathcal{O}_F(1)) \to 0.$$

$h^0(F, \mathcal{O}_F(1)) = 2$, so let $V$ be a sub vector space of $H^0(S, \mathcal{O}_S(H + rF))$ of dimension 2 that surjects onto $H^0(F, \mathcal{O}_F(1))$. This gives us a one-dimensional linear system (such things are called *pencils*), inducing:

$$|V| : S \dashrightarrow \mathbb{P}^1.$$

This is an honest-to-goodness morphism on $F$ (draw picture). The fixed components must then be components of fibers of $\pi$, and the fixed points lie on other fibers of $\pi$.

Let $\{x_1, \ldots, x_m\}$ be the set of points of $C$ where the fibers contain fixed components, or fixed points, or where the fibers are reducible. We thus get a morphism $S - \pi^{-1}\{x_1, \ldots, x_m\} \to (C - \{x_1, \ldots, x_m\}) \times \mathbb{P}^1$. This is an isomorphism. $\square$

*Remark.* If you parse this argument, you will see that the locally free sheaf you are projectivizing is $\pi_*(\mathcal{O}_S(H+iE)) = \pi_*(\mathcal{O}_S(H)) \otimes \mathcal{O}_C(i \times (\text{the point } x))$. Later today, we will see that the projectivization of a rank 2 locally free sheaf is the same as the projectivization of that sheaf twisted by a line bundle, so we could just as well have taken $\pi_*(\mathcal{O}_S(H))$, and the result is independent of $k$.

We'll now discuss geometrically ruled surfaces at some length.

**Lemma for future use.** Suppose $\pi : S \to C$ is a surjective morphism from a surface to a curve, with *connected fibers*, and $F = \sum_i n_i F_i$ is a reducible fiber of $\pi$. Then $F_i^2 < 0$ for all $i$.

(Aside: more generally, the intersection matrix of the $F_i$'s is almost negative definite. It has all negative eigenvalues except for one zero eigenvalue corresponding to $F^2 = 0$.)

*Proof.* $0 = F_i \cdot F = F_i \cdot \sum_j n_j F_j = \sum_j n_j (F_i \cdot F_j) = n_i F_i^2 + \sum_j n_j F_i \cdot F_j > n_i F_i^2$. □

**Proposition.** Let $S$ be a minimal surface, $C$ a smooth curve, $\pi : S \to C$ with generic fiber isomorphic to $\mathbb{P}^1$. Then $S$ is geometrically ruled by $\pi$, i.e. *all* fibers are isomorphic to $\mathbb{P}^1$.

*Proof.* Let $F$ be a fiber of $\pi$, so $F^2 = 0$, $F \cdot K = -2$. $F$ can't be multiple, as described earlier. So we'll show that it can't be reducible. If it were, then let $F = \sum n_i F_i$ be a reducible fiber.

$$F_i \cdot K = F_i^2 + F_i \cdot K = 2g_i - 2 \geq -2$$

from which $F_i \cdot K \geq -1$. If equality holds, then $g_i = 0$ and $F_i^2 = -1$, in which case by Castelnuovo's criterion $F_i$ is an exceptional curve; but we said that $S$ was minimal. So in fact $K \cdot F_i \geq 0$, from which $K \cdot F \geq 0$, contradicting $F \cdot K = -2$. □

**Remark: Elementary transformations.** Now is a good time to describe how to get new geometrically ruled surfaces from old ones. Geometrically: take a point on a fiber, blow it up. The strict transform has self-intersection

$$(F^{\text{strict}})^2 = (F^{\text{proper}} - E)^2 = (F^{\text{proper}})^2 + E^2 = 0 + -1,$$

so we can blow it down. This is called an *elementary transformation*. A trick we'll use later: If you blow up, then to compute the self-intersection of the proper transform, take the self-intersection of the original divisor, and subtract the multiplicity of the original curve at the point.

**Theorem (Minimal models of $C \times \mathbb{P}^1$ for $C$ irrational, i.e. not $\cong \mathbb{P}^1$).** Let $C$ be a curve not isomorphic to $\mathbb{P}^1$. The minimal models of $C \times \mathbb{P}^1$ are the geometrically ruled surfaces over $C$.

(We'll deal with the rational case soon.)

*Proof.* First, a geometrically ruled surface $\pi : S \to C$ has no exceptional curves. Reason: If there were one, where would it be? It couldn't surject onto $C$, as by the Riemann-Hurwitz formula there are no morphisms from $\mathbb{P}^1$ onto a curve of positive genus. It can't lie in a fiber, as it would then *be* a fiber, and fibers have self-intersection $0$, not $-1$.

Now let $S$ be a minimal surface, with a birational map $\phi : S \dashrightarrow C \times \mathbb{P}^1$, and let $pr_1$ be the projection of $C \times \mathbb{P}^1 \to C$. Then we have $pr_1 \circ \phi : S \dashrightarrow C$. By the elimination of indeterminacy theorem, we have

$$\begin{tikzcd} & S' \arrow[dl, "\epsilon"'] \arrow[dr, "f"] & \\ S \arrow[rr, dashed, "pr_1 \circ \phi"] & & C \end{tikzcd}$$

where $f$ is a morphism and $\epsilon$ is a combination of blow-ups $\epsilon_1 \circ \cdots \circ \epsilon_n$. Take $n$ to be the minimal number needed. We'll see that $n = 0$, and hence that the bottom rational map is actually a morphism by the previous Proposition!

If $n > 0$, then consider

$$\begin{tikzcd} & S' \arrow[dl, "\epsilon_n"'] \arrow[dr, "f"] & \\ S_{n-1} \arrow[rr, dashed] & & C \end{tikzcd}$$

Then the exceptional divisor of $\epsilon_n$ must map to a fixed point of $C$, so by the second universal property of blowing-up, the horizontal rational map is also a morphism, contradicting the minimality of $n$.

All that's left is **Exercise:** Apply hte previous Proposition to finish off the proof. $\square$

## 2. Geometrically ruled surfaces and projectivizations of rank 2 locally free sheaves

**Proposition.** Every geometrically ruled surface over $C$ is $C$-isomorphic to $\mathbb{P}_C(E)$ for some rank 2 locally free sheaf (vector bundle) over $C$. The bundles $\mathbb{P}_C(E)$ and $\mathbb{P}_C(E')$ are isomorphic (over $C$) iff there is an invertible sheaf (line bundle) $L$ on $C$ such that $E' \cong E \otimes L$.

So if we want to understand geometrically ruled surfaces, we are reduced to understanding rank 2 vector bundles. Moreover, we can twist these vector bundles by line bundles, we can reduce things farther. Define the degree of a rank 2 sheaf by $\deg E = \deg(\wedge^2 E)$. Then $\deg(E \otimes L) = \deg E + 2\deg L$, so we can restrict to the case of rank 2 invertible sheaves of degree 0 and 1 if we wanted.

*Proof.* By the Noether-Enriques theorem, a geometrically ruled surface over $C$ is locally trivial (in the Zariski topology). Thus these projective bundles are classified by $H^1(C, \mathrm{Aut}_U(U \times \mathbb{P}^1) = PGL(2, \mathcal{O}_C))$. (Warning: This group $PGL(2)$ is nonabelian!) (Sketch identification.)

From the long exact sequence for $1 \to \mathcal{O}_C^* \to GL(2, \mathcal{O}_C) \to PGL(2, \mathcal{O}_C) \to 1$, we get:

$$\mathrm{Pic}(C) \to H^1(C, GL(2, \mathcal{O}_C)) \to H^1(C, PGL(2, \mathcal{O}_C)) \to H^2(C, \mathcal{O}_C^*).$$

The term on the right is 0 for dimensional reasons (in the Zariski topology, all cohomology vanishes above the dimension) so we have:

$$\operatorname{Pic}(C) \to H^1(C, GL(2, \mathcal{O}_C)) \to H^1(C, PGL(2, \mathcal{O}_C)) \to 0.$$

The second term parametrizes rank 2 vector bundles. Now as an **exercise**, you can check that (i) The map from the second term to the third is the "take projectivization" map. (ii) The action of the first term on the second is by tensoring the vector bundle by the line bundle. $\square$

Next time we'll prove:

**Lemma: All rank 2 locally free sheaves are filtered nicely by invertible sheaves.** Suppose $E$ is a rank 2 locally free sheaf on a curve $C$. There exists an exact sequence $0 \to L \to E \to M \to 0$ with $L, M \in \operatorname{Pic} C$. Terminology: $E$ is an *extension* of $M$ by $L$.

As an immediate application, we have *Riemann-Roch for rank 2 vector bundles on a curve*:

$$\chi(E) = \chi(L) + \chi(M) = \deg(L) - g + 1 + \deg M - g + 1 = \deg E + 2(1 - g).$$

In fact, a similar lemma and similar follow-up show *Riemann-Roch for rank d vector bundles on a curve*:

$$\chi(E) = \deg E + d(1 - g).$$