MODERN ALGEBRA (MATH 210) PROBLEM SET 7

1. Prove that $\mathbb{Q}(\pi) \cong \mathbb{Q}(x)$. You may use the fact that π is transcendental.

2. Suppose q is a prime power. Show that there are $(q^3 - q)/3$ irreducible monic degree 3 polynomials in $\mathbb{F}_q[x]$. (*Hint:* Consider the elements of $\mathbb{F}_{q^3}$, and their minimal polynomials over q.) How many irreducible monic degree 12 polynomials are there in $\mathbb{F}_q[x]$?

3. Let E be the field $k(x)$. Consider the six automorphisms of E given by mapping $f(x)$ to $f(x)$, $f(1-x)$, $f(1/x)$, $f(1-1/x)$, $f(1/(1-x))$, $f(x/(x-1))$ respectively. Show that these automorphisms form a group. Let F be the fixed point field. Show that $I = (x^2 - x + 1)^3/x^2(x-1)^2 \in F$. Show that $F = k(I)$ and $[E : F] = 6$. (*Hint:* Find a sixth degree equation with coefficients in $k(I)$ satisfied by x. Use a useful theorem from class.) Hence $k(x)/k(I)$ is a Galois extension. What is its Galois group?

4. Suppose $p_1, \ldots, p_n$ are distinct prime numbers, and let $F = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$. Find the degree of the extension $F/\mathbb{Q}$. Show that $F/\mathbb{Q}$ is Galois, and find its Galois group.

5. Suppose $e_1, \ldots, e_n$ are the elementary symmetric functions in the n variables $x_1, \ldots, x_n$ over some field k, i.e. $e_1 = x_1 + \ldots x_n$, $e_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n, \ldots, e_n = x_1x_2 \ldots x_n$. Let $p_i = x_1^i + \cdots + x_n^i$ (i a positive integer). By the theorem of symmetric functions, p_i is a polynomial in the e_j. Show this explicitly as follows. We have seen that the polynomial $f(T) \in (k(x_1, \ldots, x_n))[T]$

$$f(T) = T^n - e_1T^{n-1} + e_2T^{n-2} - \cdots + (-1)^{n-1}e_{n-1}T + (-1)^n e_n$$

has roots $x_1, \ldots, x_n$. Rewrite $x_1^a f(x_1) + \cdots + x_n^a f(x_n) = 0$ in terms of $p_a, \ldots, p_{a+n}$ (a a positive integer), and show that if p_i is a polynomial in the e_j for $i < a + n$, then p_{a+n} is too. Deal also with the cases where $1 \leq i \leq n$.

6. Suppose E/F is a Galois extension of degree n with Galois group $\{\sigma_1, \ldots, \sigma_n\}$, and $a_1, \ldots, a_n$ is a basis for E over F. Show that

$$\det \begin{pmatrix} \sigma_1(a_1) & \sigma_1(a_2) & \cdots & \sigma_1(a_n) \\ \sigma_2(a_1) & \sigma_2(a_2) & \cdots & \sigma_2(a_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n(a_1) & \sigma_n(a_2) & \cdots & \sigma_n(a_n) \end{pmatrix}^2$$

is a non-zero element of F. (*Hint:* Show that it is non-zero separately. Recall the proof that if $\{\sigma_1, \ldots, \sigma_n\}$ is a group of automorphisms of a field E', and F' is the fixed field, then $[E' : F'] = n$. The proof used the fact that there could be no nontrivial solution to a certain linear equation.)

The set is due Tuesday, December 3 at 3:30 pm in Pierre Albin's mailbox.

Date: Tuesday, November 26, 2002.