MODERN ALGEBRA (MATH 210) PROBLEM SET 6

1. If q is prime, show that the automorphism group of the group $\mathbb{Z}/q$ is isomorphic to $\mathbb{Z}/(q-1)$.

2. Suppose p and q are primes such that $p|q-1$. Show the existence of a nonabelian group of order pq. (*Hint:* try a semi-direct product.)

3. If p is prime, prove that every group G of order $2p$ is either cyclic or isomorphic to D_{2p}. *Hint:* By Cauchy's theorem, G must contain an element of order p, and $\langle a \rangle$ is normal in G because it has index 2.

Problem 2 is part of the classification of groups of order pq, where $p < q$ are prime. The rest is given by: (i) If p doesn't divide $q-1$, then all groups of order pq are cyclic. (ii) If p divides $q-1$, there is only one nonabelian group of order pq. Problem 3 is a special case of (ii).

4. Suppose G is a finite group of order mn, $\gcd(m,n) = 1$. Suppose K is a normal subgroup of order m. Then K has a complement iff G has a subgroup of order n.

5. (Using semidirect products to see interesting phenomena.) Suppose G is a finite group, and you want to show that g normalizes $A \leq G$, i.e. that $g^{-1}Ag = A$. Then it suffices to show that $g^{-1}Ag \subset A$. Show that this isn't true for infinite groups (i.e. that it is possible to have $g^{-1}Ag \subsetneq A$) as follows. Let $K = \mathbb{Q}$, and $Q = \mathbb{Z} = \langle x \rangle$. Let $A \subset K$ be given by $\mathbb{Z} \subset \mathbb{Q}$. Choose a $\phi : Q \to \operatorname{Aut} K$ so that if $G = K \rtimes_\phi Q$, $xAx^{-1} \subsetneq A$.

6. Show that the splitting field of $x^3 - 3x + 1$ is degree 3 over $\mathbb{Q}$. (*Hint:* If α is a root, show that $1/(1-\alpha)$ is also a root.)

7. Show that if an irreducible cubic in $\mathbb{Q}[x]$ has two complex roots and one real root, then its splitting field is a degree 6 extension of $\mathbb{Q}$.

8. (In preparation for showing the insolvability of the quintic.) Prove that if σ is a 5-cycle and τ is a transposition in S_5, then S_5 is generated by $\{\sigma, \tau\}$.

The set is due Tuesday, November 26 at 3:30 pm in Pierre Albin's mailbox.

Date: Tuesday, November 19, 2002.