MODERN ALGEBRA (MATH 210) PROBLEM SET 4

From the text: 5.27–5.31. If you use results of earlier exercises (see 5.29), make sure to prove them!

1. Suppose F is a finite field. Find the product of the elements of $F^\times$. (Hint: try a few examples.)

2. Describe a 2-Sylow subgroup of S_8. How many are there?

3. Describe all 2-Sylow subgroups of S_4. Identify this group.

4. Do one problem from Problem Set $3\frac{1}{2}$.

5. (Dummit and Foote p. 169) For any group G define the *dual group* of G (denoted $\hat{G}$) to be the set of all homomorphisms from G into the multiplicative group of roots of unity in $\mathbb{C}$. Define a group operation in $\hat{G}$ by pointwise multiplication of functions: if χ, ψ are homomorphisms from G into the group of roots of unity then $\chi\psi$ is the homomorphism given by $(\chi\psi)(g) = \chi(g)\psi(g)$ for all $g \in G$, where the latter multiplication takes place in $\mathbb{C}$.

(a) Show that this operation on $\hat{G}$ makes $\hat{G}$ into an abelian group. [Show that the identity is the map $g \mapsto 1$ for all $g \in G$ and the inverse of $\chi \in \hat{G}$ is the map $g \mapsto \chi(g)^{-1}$.]
(b) If G is a finite abelian group, prove that $\hat{G} \cong G$. [Write G as $\langle x_1 \rangle \times \cdots \times \langle x_r \rangle$ and if n_i is the order of x_i define χ_i to be the homomorphism which sends x_i to $e^{2\pi i/n_i}$ and sends x_j to 1, for all $j \neq i$. Prove χ_i has order n_i in $\hat{G}$ and $\hat{G} = \langle \chi_1 \rangle \times \cdots \times \langle \chi_r \rangle$.]

This result is often phrased: a finite abelian group is self-dual. It implies that the lattice diagram (of subgroups) of a finite abelian group is the same when it is turned upside down. Note however that there is no *natural* isomorphism between G and its dual; the isomorphism depends on a choice of a set of generators for G. This is frequently stated in the form: a finite abelian group is *noncanonically* isomorphic to its dual.

The set is due Tuesday, November 12 at 3:30 pm in Pierre Albin's mailbox (opposite the elevator on the first floor of Building 380).

Date: Tuesday, November 5, 2002.