MODERN ALGEBRA (MATH 210) PROBLEM SET $3\frac{1}{2}$

The following problems appeared on recent quals. They aren't to be handed in; they are solely for practice. A useful fact you don't know yet: if k is a finite field, then $k[x]$ has infinitely many maximal ideals. Recall also that a ring is *Noetherian* if it contains no infinite strictly ascending chains of ideals

$$I_1 \subsetneq I_2 \subset \cdots .$$

By a remarkable Hallowe'en-related coincidence, one or two of these will re-appear on Problem Set 4.

1. (spring 2002) Let R be a commutative ring with unit.

(i) Let S be a *saturated* multiplicative set of R. This means that $1 \in S$, $0 \notin S$, and $xy \in S$ if and only if $x \in S$ and $y \in S$. Show that $R - S$ is a union of prime ideals. (Hint: If $a \in R - S$ consider ideals J with $a \in J \subset R - S$.)
(ii) An element $a \in R$ is a *zero divisor* if $ab = 0$ for some $b \neq 0$. Apply (i) to show that the set of zero divisors is a union of prime ideals of R.

2. (fall 2000)

(a) If A is a commutative ring, show that the polynomial ring $A[X]$ contains infinitely many distinct maximal ideals.
(b) Describe all maximal ideals in the ring of formal power series $\mathbb{Z}[[X]]$.

3. (fall 1999) Suppose V is a commutative ring such that for all $a, b \in V$ either $a|b$ or $b|a$.

(i) Prove that if I and J are two ideals in V then $I \subset J$ or $J \subset I$.
(ii) Prove that any finitely generated ideal of V is principal, that is, generated by a single element.
(iii) Prove that if V is a Noetherian valuation ring, then there exists an element $t \in V$ such that any proper nonzero ideal of V is (t^n) for some whole number $n \geq 1$

4. (fall 1998)

(a) Suppose that A is a Noetherian domain. Show that every nonzero noninvertible element of A can be written as a finite product of irreducible elements.
(b) Give an example of a Noetherian integral domain which is not a unique factorization domain.

Date: Hallowe'en 2002.