MODERN ALGEBRA (MATH 210) PROBLEM SET 2

From the text: 2.69, 2.76, 2.98, 2.103. (For the last, and for future reference, read and understand the proof of Burnside's Lemma, Theorem 2.113. Caution: the answer given in 2.103(ii) is incorrect!)

A1. This exercise shows that for $n \neq 6$, every automorphism of S_n is inner. Fix an integer $n \geq 2$ with $n \neq 6$.

(a) Prove that the automorphism group of a group G permutes the conjugacy classes of G, i.e. for each $\sigma \in \mathrm{Aut}(G)$ and each conjugacy class $\mathcal{K}$ of G the set $\sigma(\mathcal{K})$ is also a conjugacy class of G.

(b) Let $\mathcal{K}$ be the conjugacy class of transpositions in S_n and let $\mathcal{K}'$ be the conjugacy class of any element of order 2 in S_n that is not a transposition. Prove that $|\mathcal{K}| \neq |\mathcal{K}'|$. Deduce that any automorphism of S_n sends transpositions to transpositions.

(c) Prove that for each $\sigma \in \mathrm{Aut}(S_n)$

$$\sigma : (12) \mapsto (ab_2), \qquad \sigma : (13) \mapsto (ab_3), \ldots, \sigma : (1n) \mapsto (ab_n)$$

for some distinct integers $a, b_2, b_3, \ldots, b_n \in \{1, 2, \ldots, n\}$.

(d) Show that (12),(13), ... , $(1n)$ generate S_n and deduce that any automorphism of S_n is uniquely determined by its action on these elements. Use (c) to show that S_n has at most $n!$ automorphisms and conclude that $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n)$ for $n \neq 6$.

A2. We now show that $\mathrm{Inn}(S_6)$ is of index at most 2 in $\mathrm{Aut}(S_6)$. Let $\mathcal{K}$ be the conjugacy class of transpositions in S_6 and let $\mathcal{K}'$ be the conjugacy class of any element of order 2 in S_6 that is not a transposition. Prove that $|\mathcal{K}| \neq |\mathcal{K}'|$ unless $\mathcal{K}'$ is the conjugacy class of products of three disjoint transpositions. Deduce that $\mathrm{Aut}(S_6)$ has a subgroup of index at most 2 which sends transpositions to transpositions. Then prove that $|\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)| \leq 2$.

A3. Finally, we exhibit an outer automorphism of S_6. (There are other, more beautiful, descriptions.) Let $t_1' = (12)(34)(56)$, $t_2' = (14)(25)(36)$, $t_3' = (13)(24)(56)$, $t_4' = (12)(36)(45)$, $t_5' = (14)(23)(56)$. Show that t_1', ... , t_5' satisfy the following relations:

- $(t_i')^2 = e$ for all i;
- $(t_i' t_j')^2 = e$ for all i and j with $|i - j| \geq 2$;
- $(t_i' t_j')^3 = e$ for all i and j with $|i - j| = 1$.

Use this to show that the map $(i(i+1)) \mapsto t_i'$ gives an automorphism of S_6. (In the process, you will likely have to show that the relations above define S_6. Your argument will also presumably prove the obvious generalization to S_n.)

Date: Tuesday, October 15, 2002.

B1. If there exists a chain of subgroups $G_1 \leq G_2 \leq \cdots \leq G$ such that $G = \cup_{i=1}^{\infty} G_i$ and each G_i is simple, then G is simple. (Note that G need not be finite!)

B2. (a) Let Ω be an infinite set. Let D the subgroup of S_Ω consisting of permutations which move only a finite number of elements of Ω and let A be the set of all elements $\sigma \in D$ such that σ acts as an even permutation on the (finite) set of points it moves. Prove that A is an infinite simple group.

(b) Prove that if $H \neq \{e\}$ is a normal subgroup of S_Ω, then H contains A, i.e. A is the unique nontrivial minimal normal subgroup of S_Ω.

C. For any finite group P, let $d(P)$ be the minimum number of generators of P (so, for example, $d(P) = 1$ iff P is a nontrivial cyclic group). Let $m(P)$ be the maximum of the integers $d(A)$ as A runs over all *abelian* subgroups of P. Define

$$J(P) = \langle A : A \text{ is an abelian subgroup of } P \text{ with } d(A) = m(P) \rangle.$$

($J(P)$ is called the *Thompson subgroup* of P. It plays a pivotal role in the study of finite groups, and in particular the classification of finite simple groups.)

(a) Prove that $J(P)$ is preserved by all automorphisms of P. (This is the definition of a *characteristic subgroup*.) Hence show that $J(P)$ is normal.

(b) For both $P = Q_8$ and D_8, list all abelian subgroups A of P that satisfy $d(A) = m(P)$. In both cases show that $J(P) = P$.

(c) Prove that if $Q \leq P$ and $J(P)$ is a subgroup of Q, then $J(P) = J(Q)$. Deduce that if P is a subgroup (not necessarily normal) of the finite group G and $J(P)$ is contained in some subgroup Q of P such that Q is normal in G, then $J(P)$ is normal in G as well.

Problems A1–C are from Dummit and Foote.

The set is due Tuesday, October 22 at 3:30 pm in Pierre Albin's mailbox (opposite the elevator on the first floor of Building 380).