MODERN ALGEBRA (MATH 210) EXAM PRACTICE PROBLEMS AND GALOIS THEORY PRACTICE PROBLEMS

The final exam will have roughly 9 problems, with roughly 2 on groups, 2 on rings, 1 on Jordan-Holder, 1 on semidirect products, and 3 on Galois theory.

1. Practice problems for the exam

P1. Show that the rotation group of the cube is isomorphic to S_4.

P2. If A and B are subgroups of finite index in a group G, and the indices of A and B in G are relatively prime, show that $G = AB$.

P3. Suppose p is an odd prime.

(a) Show that exactly half of $(\mathbb{Z}/p\mathbb{Z})^* = \{1, 2, \ldots, p-1\}$ are squares modulo p. (*Hint:* consider the structure of the group $(\mathbb{Z}/p\mathbb{Z})^*$.)
(b) Prove that $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$ for all $a \in (\mathbb{Z}/p\mathbb{Z})^*$.
(c) Show that $a^{(p-1)/2} \equiv 1 \pmod{p}$ if and only if a is a perfect square in $(\mathbb{Z}/p\mathbb{Z})^*$.
(d) Show that if neither a nor b are perfect squares modulo p, then ab is a perfect square modulo p.

P4. Are the following ideals prime in $\mathbb{C}[x, y]$?

(a) $(x, y-1)$,
(b) (x, y^2),
(c) $(y - x^2, y - 1)$.

P5. If $\alpha \in E$ an extension of F, and $f(x), g(x) \in F[x]$ with $f(\alpha) = g(\alpha) = 0$, and f is irreducible, show that $f(x)$ is a factor of $g(x)$.

P6. Let p and q be distinct primes.

(i) Prove that every group of order pq is solvable. (*Hint:* show that the group can't be simple.)
(ii) Prove that every group G of order p^2q is solvable. (*Hint*: Show that G can't be simple.)

P7. Suppose $f(x) \in F[x^p]$ is irreducible of positive degree. Show that $f(x)$ is *not* separable.

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P8.

(a) Suppose the splitting field E of a cubic $f(x)$ over $\mathbb{Q}$ has Galois group S_3, and $f(x)$ has roots x_1, x_2, x_3. Show that E^{A_3} is generated (over $\mathbb{Q}$) by $\Delta = (x_1-x_2)(x_1-x_3)(x_2-x_3)$.
(b) Show that the splitting field E of a cubic $f(x) = x^3 + ax^2 + bx + c$ over $\mathbb{Q}$ has Galois group $\mathbb{Z}/3\mathbb{Z}$ if and only if $\sqrt{\Delta} \in \mathbb{Q}$, where $\Delta := a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc$. (If E has roots x_1, x_2, x_3, then you may use the fact that $[(x_1-x_2)(x_1-x_3)(x_2-x_3)]^2 = \Delta$, which you could have computed with some effort from a problem on the last problem set.)

P9. Let E be the splitting field of

$$f(x) = (x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x^1 + 1$$

over $\mathbb{Q}$. Let ζ be a zero of $f(x)$, i.e. a primitive seventh root of 1.

(a) Show that $f(x)$ is irreducible. (*Hint:* consider $f(y + 1)$ and use Eisenstein.)
(b) Find the degree of the extension $E/\mathbb{Q}$.
(c) Show that the Galois group of $E/\mathbb{Q}$ is cyclic, and find an explicit generator.
(d) Let $\beta = \zeta + \zeta^2 + \zeta^4$. Show that the intermediate field $\mathbb{Q}(\beta)$ is actually $\mathbb{Q}(\sqrt{-7})$. (*Hint:* first show that $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$ by finding a linear dependence over $\mathbb{Q}$ among $\{1, \beta, \beta^2\}$.)
(e) Let $\gamma_q = \zeta + \zeta^q$. Find a q such that $\mathbb{Q}(\gamma_q)$ is a degree 3 extension of $\mathbb{Q}$. (*Possible hint:* use (c).) Is this extension Galois?

2. Galois theory practice problems

G1. Suppose G is the Galois group of the Galois extension E/F, and $H < G$ is a subgroup (not necessarily normal) with α conjugates. Find $|\operatorname{Aut}(H/k)|$ in terms of α, $|G|$, $|H|$.

G2. Describe the splitting field of $x^3 - 2$ over $\mathbb{Q}$, and all intermediate fields.

G3. Suppose F is characteristic 2, and E/F is a degree 2 extension with $E = F(\alpha)$, and α satisfying $T^2 + bT + c = 0$. What is the condition on b and c such that E/F is Galois?

G4. Show that the polynomial $t^5 - 4t + 2$ is not soluble by radicals.