

# MODERN ALGEBRA (MATH 210) EXAM PRACTICE PROBLEMS AND GALOIS THEORY PRACTICE PROBLEMS

*The final exam will have roughly 9 problems, with roughly 2 on groups, 2 on rings, 1 on Jordan-Holder, 1 on semidirect products, and 3 on Galois theory.*

## 1. PRACTICE PROBLEMS FOR THE EXAM

**P1.** Show that the rotation group of the cube is isomorphic to  $S_4$ .

**P2.** If  $A$  and  $B$  are subgroups of finite index in a group  $G$ , and the indices of  $A$  and  $B$  in  $G$  are relatively prime, show that  $G = AB$ .

**P3.** Suppose  $p$  is an odd prime.

- (a) Show that exactly half of  $(\mathbb{Z}/p\mathbb{Z})^* = \{1, 2, \dots, p-1\}$  are squares modulo  $p$ . (*Hint: consider the structure of the group  $(\mathbb{Z}/p\mathbb{Z})^*$ .)*
- (b) Prove that  $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$  for all  $a \in (\mathbb{Z}/p\mathbb{Z})^*$ .
- (c) Show that  $a^{(p-1)/2} \equiv 1 \pmod{p}$  if and only if  $a$  is a perfect square in  $(\mathbb{Z}/p\mathbb{Z})^*$ .
- (d) Show that if neither  $a$  nor  $b$  are perfect squares modulo  $p$ , then  $ab$  is a perfect square modulo  $p$ .

**P4.** Are the following ideals prime in  $\mathbb{C}[x, y]$ ?

- (a)  $(x, y-1)$ ,
- (b)  $(x, y^2)$ ,
- (c)  $(y-x^2, y-1)$ .

**P5.** If  $\alpha \in E$  an extension of  $F$ , and  $f(x), g(x) \in F[x]$  with  $f(\alpha) = g(\alpha) = 0$ , and  $f$  is irreducible, show that  $f(x)$  is a factor of  $g(x)$ .

**P6.** Let  $p$  and  $q$  be distinct primes.

- (i) Prove that every group of order  $pq$  is solvable. (*Hint: show that the group can't be simple.*)
- (ii) Prove that every group  $G$  of order  $p^2q$  is solvable. (*Hint: Show that  $G$  can't be simple.*)

**P7.** Suppose  $f(x) \in F[x^p]$  is irreducible of positive degree. Show that  $f(x)$  is *not* separable.

P8.

- (a) Suppose the splitting field  $E$  of a cubic  $f(x)$  over  $\mathbb{Q}$  has Galois group  $S_3$ , and  $f(x)$  has roots  $x_1, x_2, x_3$ . Show that  $E^{A_3}$  is generated (over  $\mathbb{Q}$ ) by  $\Delta = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ .
- (b) Show that the splitting field  $E$  of a cubic  $f(x) = x^3 + ax^2 + bx + c$  over  $\mathbb{Q}$  has Galois group  $\mathbb{Z}/3\mathbb{Z}$  if and only if  $\sqrt{\Delta} \in \mathbb{Q}$ , where  $\Delta := a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc$ . (If  $E$  has roots  $x_1, x_2, x_3$ , then you may use the fact that  $[(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]^2 = \Delta$ , which you could have computed with some effort from a problem on the last problem set.)

P9. Let  $E$  be the splitting field of

$$f(x) = (x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

over  $\mathbb{Q}$ . Let  $\zeta$  be a zero of  $f(x)$ , i.e. a primitive seventh root of 1.

- (a) Show that  $f(x)$  is irreducible. (*Hint*: consider  $f(y + 1)$  and use Eisenstein.)
- (b) Find the degree of the extension  $E/\mathbb{Q}$ .
- (c) Show that the Galois group of  $E/\mathbb{Q}$  is cyclic, and find an explicit generator.
- (d) Let  $\beta = \zeta + \zeta^2 + \zeta^4$ . Show that the intermediate field  $\mathbb{Q}(\beta)$  is actually  $\mathbb{Q}(\sqrt{-7})$ . (*Hint*: first show that  $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$  by finding a linear dependence over  $\mathbb{Q}$  among  $\{1, \beta, \beta^2\}$ .)
- (e) Let  $\gamma_q = \zeta + \zeta^q$ . Find a  $q$  such that  $\mathbb{Q}(\gamma_q)$  is a degree 3 extension of  $\mathbb{Q}$ . (*Possible hint*: use (c).) Is this extension Galois?

## 2. GALOIS THEORY PRACTICE PROBLEMS

G1. Suppose  $G$  is the Galois group of the Galois extension  $E/F$ , and  $H < G$  is a subgroup (not necessarily normal) with  $\alpha$  conjugates. Find  $|\text{Aut}(H/k)|$  in terms of  $\alpha$ ,  $|G|$ ,  $|H|$ .

G2. Describe the splitting field of  $x^3 - 2$  over  $\mathbb{Q}$ , and all intermediate fields.

G3. Suppose  $F$  is characteristic 2, and  $E/F$  is a degree 2 extension with  $E = F(\alpha)$ , and  $\alpha$  satisfying  $T^2 + bT + c = 0$ . What is the condition on  $b$  and  $c$  such that  $E/F$  is Galois?

G4. Show that the polynomial  $t^5 - 4t + 2$  is not soluble by radicals.