THE FUNDAMENTAL THEOREM OF GALOIS THEORY

Important Theorem. If E/k is a finite extension, then the following statements are equivalent.

(i) E is a splitting field of some separable polynomial $f(x) \in k[x]$.
(ii) $k = E^G$, where G is the group of automorphisms of E fixing k (i.e. E/k is Galois).
(iii) Every irreducible $p(x) \in k[x]$ having one root in E is separable and splits in $E[x]$.

Fundamental Theorem of Galois theory (Rotman p. 228). Let E/k be a finite Galois extension with Galois group $G = \mathrm{Gal}(E/k)$.

(i) The function

$$\gamma : \text{intermediate fields of } E/k \to \text{subgroups of } \mathrm{Gal}(E/k),$$

defined by $\gamma : F \mapsto \mathrm{Gal}(E/F)$, is an order-reversing bijection with inverse

$$\delta : \text{subgroups of } \mathrm{Gal}(E/k) \to \text{intermediate fields of } (E/k),$$

given by $\delta : H \mapsto E^H$.

(ii) For every intermediate field $E/B/k$, $E^{\mathrm{Gal}(E/B)} = B$. For every subgroup $H \subset \mathrm{Gal}(E/k)$, $\mathrm{Gal}(E/E^H) = H$.

(iii) For any two subgroups $H, K \subset \mathrm{Gal}(E/k)$,

$$E^{\langle H,K\rangle} = E^H \cap E^K$$

(where $\langle H, K\rangle$ means the subgroup generated by H, K) and

$$E^{H\cap K} = \langle E^H, E^K\rangle$$

(where $\langle E^H, E^K\rangle$ means the subfield generated by E^H, E^K, called the *compositum*). For any two intermediate fields B and C of E/k,

$$\mathrm{Gal}(E/\langle B,C\rangle) = \mathrm{Gal}(E/B)\cap\mathrm{Gal}(E/C) \qquad \text{and} \qquad \mathrm{Gal}(E/(B\cap C)) = \langle \mathrm{Gal}(E/B), \mathrm{Gal}(E/C)\rangle.$$

(iv) For every intermediate field B between k and E, $[B : k] = [G : \mathrm{Gal}(E/B)]$. (The left is the degree of a field extension, and the right is an index of a subgroup!) For every subgroup H of $\mathrm{Gal}(E/k)$, $[G : H] = [E^H : k]$.

(v) If B is an intermediate field, then B/k is Galois iff $\mathrm{Gal}(E/B)$ is a normal subgroup of $\mathrm{Gal}(E/k)$. In this case, $\mathrm{Gal}(B/k) = \mathrm{Gal}(E/k)/\mathrm{Gal}(E/B)$.

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