18.014 QUIZ III SOLUTIONS

0. (1 point) Write your name and your recitation instructor's name on the first page of your solutions.

Solution. The class did very well on this problem, so the solution is omitted.

1. (27 points) Evaluate:

(a) $\int \frac{dx}{x^3+x^2}$.
(b) $\int_e^{e^2} \frac{dx}{x \ln^2 x}$.
(c) $\int \frac{x^3 dx}{\sqrt{1-x^2}}$.

Solution.

(a) Note that $\frac{1}{x^3+x^2} = \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}$ (by partial fractions). Hence

$$\begin{aligned}\int \frac{dx}{x^3+x^2} &= \int \left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}\right) dx \\ &= -1/x - \ln|x| + \ln|x+1| + C.\end{aligned}$$

One should be careful to only apply this formula on intervals not containing $x = 0$ and $x = -1$.

(b) Substitute $u = \ln x$ to get

$$\int_1^2 \frac{du}{u^2} = \left(-\frac{1}{u}\right)_1^2 = 1/2.$$

(c) Substitute $x = \sin\theta$ $(-\pi/2 \leq \theta \leq \pi/2)$ to get

$$\begin{aligned}\int \sin^3\theta d\theta &= \int \sin\theta(1-\cos^2\theta)d\theta \\ &= \int \sin\theta d\theta - \int \sin\theta\cos^2\theta d\theta \\ &= -\cos\theta + \frac{1}{3}\cos^3\theta + C\end{aligned}$$

As $\cos\theta = \sqrt{1-x^2}$ (why not $-\sqrt{1-x^2}$?), we get

$$-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C.$$

(What would happen if we chose θ in a different range, e.g. $\pi/2 \leq \theta \leq 3\pi/2$?)

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Remark. Substituting $x = \cos\theta$ also works. Also, substituting $u = 1 - x^2$ turns out to be the simplest attack (do you see why?), although this isn't clear from the outset.

2. (15 points) Define $\lim_{x\to+\infty} f(x) = 3$ and $\lim_{t\to 0^+} f(1/t) = 3$. Explain why one equality is true if the other is.

Solution. $\lim_{x\to+\infty} f(x) = 3$ means: For every $\epsilon > 0$, there is some $N > 0$ so that when $x > N$, $|f(x) - 3| < \epsilon$.

$\lim_{t\to 0^+} f(1/t) = 3$ means: For every $\epsilon > 0$, there is some $\delta > 0$ so that when $0 < t < \delta$, $|f(1/t) - 3| < \epsilon$.

By letting $\delta = 1/N$, the first defintion becomes the second (and vice versa); just take $x = 1/t$.

3. (18 points) Suppose you use the first two nonzero terms of the Taylor polynomial for $\cos x$ to compute the integral

$$\int_0^{1/2} \cos(x^2)dx.$$

(a) What answer do you get? (Leave as a sum of fractions.)
(b) Is your answer greater or less than the actual value?
(c) Obtain an upper bound on the error. (Leave in terms of fractions.)

Solution. Lagrange's form of the remainder in Taylor's theorem gives, when $x > 0$:

$$\cos x = 1 - \frac{x^2}{2} + (\cos c)\frac{x^4}{24}$$

where $0 < c < x$. Hence

$$\cos x^2 = 1 - \frac{x^4}{2} + (\cos c)\frac{x^8}{24} \tag{1}$$

where $0 < c < x^2$. (Other variations are possible. For example, $\cos x = 1 - x^2/2 + (\sin c)x^3/6$ where $0 < c < x$.)

(a)

$$\int_0^{1/2} \cos(x^2)dx \approx \int_0^{1/2} \left(1 - x^4/2\right) dx = \left(x - x^5/10\right)_0^{1/2} = 1/2 - 1/320.$$

(b) When $0 \le x \le 1/2$, $0 < c < x^2 \le 1/4$. Hence $\cos c > 0$, as $1/4 < \pi/2$. (Recall, for example, that we've shown that $\pi \ge 3$, see Notes L.) By (1), $\cos(x^2) \ge 1 - x^4/2$ for $0 \le x \le 1/2$, so the answer to (a) is less than the actual value.

(c) As $\cos c \le 1$, the error is at most

$$\int_0^{1/2} \frac{x^8}{24}dx = \left(\frac{x^9}{9\times 24}\right)_0^{1/2} = \frac{1}{2^9\times 9\times 24}.$$

4. (12 points) There is a positive integer m such that

$$\lim_{x\to 0} \frac{\sin(2x^3) - 2x^3}{x^m}$$

is finite and nonzero. What is m, and what is the limit L?

Solution. By Lagrange's form for the remainder in Taylor's formula,

$$\sin(2x^3) = (2x^3) - \frac{(2x^3)^3}{3!} + B(x)(2x^3)^4$$

where $B(x)$ is a bounded function for $|x| < 1$ (or any other finite interval containing 0). Hence

$$\frac{\sin(2x^3) - 2x^3}{x^m} = -\frac{4}{3}x^{9-m} + 2^4 B(x) x^{12-m}.$$

Thus the limit is finite and non-zero if and only if $m = 9$; in this case, the limit is

$$\lim_{x\to 0} \left(-4/3 + 2^4 B(x) x^3\right) = -4/3.$$

Question: If you used l'Hopital's rule instead, how many iterations would it take?

5. (27 points) Evaluate:

(a) $\lim_{x\to 0} \frac{\sin^2(ax)}{1-\cos(bx)}$.

(b) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

(c) $\lim_{x\to 0+} \frac{e^{-1/x}}{x}$.

Solution.

(a) (Here we need to assume that $b \neq 0$. Do you see why?) L'Hopital's rule or Lagrange's formula, directly applied, would work. For the sake of variety, we present a different solution. Easy applications of l'Hopital or Lagrange gives:

$$\lim_{x\to 0} \frac{\sin(ax)}{x} = a$$

and

$$\lim_{x\to 0} \frac{x^2}{1-\cos(bx)} = 2/b^2.$$

Hence:

$$\lim_{x\to 0} \frac{\sin^2(ax)}{1-\cos(bx)} = \lim_{x\to 0} \left(\frac{\sin(ax)}{x} \cdot \frac{\sin(ax)}{x} \cdot \frac{x^2}{1-\cos(bx)} = a \cdot a \cdot (2/b^2)\right) = 2a^2/b^2.$$

(b) We use l'Hopital's rule twice:

$$\begin{aligned}
\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) &= \lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}\\
&= \lim_{x\to 1^+}\frac{1-1/x}{1-1/x+\ln x}\\
&= \lim_{x\to 1^+}\frac{1/x^2}{1/x^2+1/x}\\
&= \lim_{x\to 1^+}\frac{1}{x+1}\\
&= 1/2.
\end{aligned}$$

Lagrange's formula could also be used.

(c) Using problem 2 and l'Hopital's rule,

$$\lim_{x\to 0^+}\frac{e^{1/x}}{x}=\lim_{n\to+\infty}\frac{n}{e^n}=\lim_{n\to+\infty}\frac{1}{e^n}=0.$$

Another possibility: use

$$\frac{n}{e^n}<\frac{n}{(1+1)^n}<\frac{n}{1+n+n(n-1)/2}.$$

(Quite a few other methods also work; many other methods *don't* work!)

Challenge problem from practice quiz. Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 111\cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

(This problem appeared on the Putnam competition. Can you guess which year?)

Solution.

Write $N=(10^{1998}-1)/9$. Then

$$\sqrt{N}=\frac{10^{999}}{3}\sqrt{1-10^{-1998}}=\frac{10^{999}}{3}(1-\frac{1}{2}10^{-1998}+r),$$

where r can be bounded using Taylor's theorem with remainder (Lagrange's form). Recall that the theorem says that $f(x)=f(0)+xf'(0)+\frac{x^2}{2}f''(c)$ for some c between 0 and x. For $f(x)=\sqrt{1+x}$, $f''(x)=-\frac{1}{4}(1+x)^{-3/2}$; therefore for $x=-10^{-1998}$, $|r|<10^{-3996}/8$.

Now the digits after the decimal point of $10^{999}/3$ are given by $.3333\ldots$, while the digits after the decimal point of $\frac{1}{6}10^{-999}$ are given by $.00000\ldots1666666\ldots$. It follows that the first 1000 digits of $\sqrt{N}$ are given by $.33333\ldots3331$; in particular, the thousandth digit is 1.

(Side question: What are the next thousand digits?)