18.014 QUIZ II SOLUTIONS

1. (24 points) Assume f is defined on the interval $[a, b]$.

(a) State the extreme value theorem for f.
(b) State the mean value theorem for f.
(c) State the first fundamental theorem of calculus for f (the one about the derivative of the integral).

Make sure you include the hypotheses for each theorem.

Solution.

(a) Assume f is continuous on $[a, b]$. Then there exist points c and d in $[a, b]$ such that $f(c) = \sup f$ and $f(d) = \sup f$. (In other words, f has a maximum and minimum on $[a, b]$.)

(b) Assume that f is continuous on $[a, b]$ and has a derivative at each point of the open interval (a, b). Then there is at least one interior point c of (a, b) for which $f(b) - f(a) = f'(c)(b - a)$.

(c) Assume that f is integrable on $[a, b]$, and let c be a point of $[a, b]$. Let $A(x) = \int_c^x f(t)dt$. If f is continuous at the point x_0 of $[a, b]$, then $A'(x_0)$ exists and $A'(x_0) = f(x_0)$.

2. (16 points) Compute the following limit; state what limit theorems you are using.

$$\lim_{h\to 0} \frac{(h+2)^3 - 8}{h(h-2)}.$$

Solution. Notice that for $h \neq 0, 2$,

$$\frac{(h+2)^3 - 8}{h(h-2)} = \frac{h^3 + 6h^2 + 18h}{h(h-2)} = \frac{h^2 + 6h + 18}{h - 2}.$$

We may replace the function $\frac{(h+2)^3-8}{h(h-2)}$ in the question by $\frac{h^2+6h+18}{h-2}$ as they have the same values away from 0 and 2, and the limit depends only on the values away from these two numbers. Polynomials are continuous, so the limit as $h \to 0$ of $h^2 + 6h + 18$ is 18, and the limit of $h - 2$ is -2. As $\lim_{h\to 0}(h - 2) \neq 0$,

$$\lim_{h\to 0} \frac{h^2 + 6h + 18}{h - 2} = \frac{\lim_{h\to 0}(h^2 + 6h + 18)}{\lim_{h\to 0}(h - 2)} = \frac{18}{-2} = -9.$$

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3. (24 points) Find $f'(x)$ if

(a)

$$f(x) = \int_{x^2}^{x^3} \frac{1}{1+t^4} dt.$$

(b)

$$f(x) = \sqrt{x^3 + 5\sqrt{x+1}}.$$

(c)

$$f(x) = \sin^2(\cos^2 x).$$

Answers. (a)

$$\frac{3x^2}{1+x^{12}} - \frac{2x}{1+x^8}.$$

(b)

$$\frac{1}{2} \cdot \frac{1}{\sqrt{x^3 + 5\sqrt{x+1}}} \cdot \left(3x^2 + \frac{5}{2\sqrt{x+1}}\right)$$

(which can be simplified further).

(c)

$$2\sin(\cos^2 x)\cos(\cos^2 x) \times 2\cos x(-\sin x) = -4\sin(\cos^2 x)\cos(\cos^2 x)\cos(x)\sin(x)$$

(which can be simplified further).

4. (16 points) Let $f(x)$ be continuous for all x except $x = 2$. Let

$$g(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ x^2 + 1 & \text{for } x < 0. \end{cases}$$

For what values of x can you be sure that the function $h(x) = f(g(x))$ is continuous?

Solution. A short calculation shows that if $x \neq \sqrt{2}, -1$, then $g(x) \neq 2$. If $x \neq \sqrt{2}, 0, -1$, then g is continuous at x and f is continuous at $g(x)$, so $f(g(x))$ is continuous at x. Thus by the theorem on the composition of continuous functions, we can be sure that h is continuous at all values except 0, $\sqrt{2}$ and -1.

We can't be sure that $f(g(x))$ is continuous at these three values. For example, if $f(x) = x$, then $f(g(x))$ is discontinuous at 0. If

$$f(x) = \begin{cases} 1 & \text{for } x = 2 \\ 0 & \text{otherwise,} \end{cases}$$

then

$$f(g(x)) = \begin{cases} 1 & \text{for } x = -1 \text{ or } \sqrt{2} \\ 0 & \text{otherwise,} \end{cases}$$

so $f(g(x))$ is discontinuous at -1 and $\sqrt{2}$.

Further question: Must $f(g(x))$ be discontinuous at these three values?

5. (20 points) The following table was computed for the strictly increasing function f and its first two derivatives. (Assume f' and f'' exist for all x.)

x	$f(x)$	$f'(x)$	$f''(x)$
0	-2	3	-2
1	0	3/2	-1/2
2	1	1	0

Let g be the inverse function to f. Find the values of $g(0)$, $g(1)$, $g'(0)$, and $g''(0)$.

Solution. The first two are immediate from the table: $g(0) = 1$, $g(1) = 2$. As

$$g'(x) = 1/f'(g(x)), \tag{1}$$

$g'(0) = 1/f'(g(0)) = 1/f'(1) = 2/3$. Differentiating (1) gives

$$g''(x) = -\frac{1}{f'(g(x))^2} \cdot f''(g(x))g'(x) = -\frac{1}{f'(g(x))^3} \cdot f''(g(x))$$

so $g''(0) = -(1/(3/2)^3)(-1/2) = 4/27$.