18.014 QUIZ I SOLUTIONS

The following are (fairly complete) sketches. If you have any questions, please ask!

1. (16 points)

(a) State the triangle inequality for $|a+b|$.
(b) Show that $|x| - |y| \leq |x-y|$ for all x, y.

Solution. (a) For any real numbers a and b, $|a+b| \geq |a| + |b|$. (b) Substitute $a = y$ and $b = x - y$ in (a), and rearrange.

2. (16 points) State the Riemann condition for the existence of the integral $\int_a^b f$, where f is a function on $[a,b]$.

Solution. Suppose f is defined on $[a,b]$. Then f is integrable on $[a,b]$ if and only if given any $\epsilon > 0$, there exist, correspondingly, step functions s and t, with $s \leq f \leq t$ on $[a,b]$, such that

$$\int_a^b t - \int_a^b s < \epsilon.$$

Warning: it is essential to say something logically equivalent to this, and not, for example, that there exist step functions s and t with $s \leq f \leq t$ on $[a,b]$ such that $\int_a^b t - \int_a^b s < \epsilon$, where ϵ is any positive number. Can you see why this is not the same thing?

3. (16 points) Evaluate $\int_{-1}^2 x^2[x]dx$, where $[\cdot]$ denotes the "greatest integer" function.

Solution.

$$\begin{aligned}
\int_{-1}^2 x^2[x]dx &= \int_{-1}^0 x^2[x]dx + \int_0^1 x^2[x]dx + \int_1^2 x^2[x]dx \\
&= \int_{-1}^0 (-x^2)dx + \int_0^1 0dx + \int_1^2 x^2 dx \\
&= \left(-x^3/3\right)\Big|_{-1}^0 + 0 + \left(x^3/3\right)\Big|_1^2 \\
&= -1/3 + 7/3 \\
&= 2.
\end{aligned}$$

(You don't need to give the names of the properties used; but note that every line is something that we've seen.)

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4. (16 points) Suppose that $\int_0^1 \frac{x}{x^6+1}dx = a$ and $\int_0^2 \frac{x}{x^6+1}dx = b$. Express $\int_{-2}^{-1} \frac{3x}{x^6+1}$ in terms of a and b.

Solution.

$$\begin{aligned}
\int_{-2}^{-1} \frac{3x}{x^6+1} &= \int_0^{-1} \frac{3x}{x^6+1} - \int_0^{-2} \frac{3x}{x^6+1} \\
&= 3\int_0^{-1} \frac{x}{x^6+1} - 3\int_0^{-2} \frac{x}{x^6+1} \\
&= 3\int_1^0 \frac{(-x)}{(-x)^6+1} - 3\int_2^0 \frac{(-x)}{(-x)^6+1} \\
&= -3\int_1^0 \frac{x}{x^6+1} + 3\int_2^0 \frac{x}{x^6+1} \\
&= 3\int_0^1 \frac{x}{x^6+1} - 3\int_0^2 \frac{x}{x^6+1} \\
&= 3a - 3b
\end{aligned}$$

The most common answer was $3b - 3a$. There is a quick way of seeing that this can't be correct: can you see why $3b-3a$ must be positive, and why $\int_{-2}^{-1} \frac{3x}{x^6+1}$ must be negative?

5. (16 points) Consider the solid in three-space that lies above $z = 0$, such that the cross-section for given z is a square with sides parallel to the x and y axes having as left edge the line segment connecting the point $(z, 0)$ on the x-axis to the point (z, z^3) on the curve $y = x^3$. Find the volume of the portion of the solid between $z = 0$ and $z = a$, where $a > 0$.

Solution. The cross-section of the solid for given z (between 0 and a) is a square of side length z^3, hence of area z^6. Thus the volume is

$$\int_0^a z^6 dz = a^7/7.$$

6. (20 points) Suppose x and y are real numbers with $x < y$.

(a) If $y - x > 1$, show that there is an integer z such that $x < z < y$. (You may use standard properties of the integers. If you use the well-ordering principle, the Archimedean property, or the principle of induction, mention the fact that you are using it.)

(b) Even if $y - x$ is not greater than 1, show that there is a rational number r such that $x < r < y$. (Hint: Why is there a positive integer n such that $y - x > 1/n$? Then consider $nx < ny$ instead of $x < y$.)

Solution.

(a) *Solution using the well-ordering principle.*

We use a useful lemma: Any *nonempty* set S of integers *bounded below* has a minimal element.

Proof of lemma: let b be the lower bound. Then there exists an integer n less than b. Let $S' = \{x | x + n \in S\}$. Then S' is a set of integers that are positive (as if $x \in S'$, then $x + n \geq b$, from which $x \geq b - n > 0$), and S' is nonempty. By the well-ordering principle, S' has a minimal element. Then $\min(S') + n$ is a minimal element of S.

We can now solve the problem. Let S be the set of integers greater than x. It is nonempty (there is an integer greater than x) and bounded below (by x), so it has a minimal element; call it z. Then $z - x \leq 1$. As $y - z = (y - x) - (z - x) \geq y - x - 1 > 0$, $y > z$, and we're done.

Solution, not using the well-ordering principle. The set of integers $\mathbb{Z}$ consists of positive integers, zero, and negatives of positive integers. We split the proof into cases.

Case 1: $x \geq 1$. Let $S = \{a \in P | a \leq x\}$. S has at least one element, 1, and is bounded above by $x + 1$. Therefore by the least upper bound axiom there is a real number $s = \sup S$. Since $s - 1 < s$, $s - 1$ is not an upper bound of S. Therefore there exists $k \in S$ such that $k > s - 1$. Then $k + 1 > s = \sup S$ and therefore $k + 1 \notin S$. Since $k \in \mathbb{Z}^+$, it belongs to every inductive set. Therefore $k + 1$ belongs to every inductive set, and $k + 1 \in \mathbb{Z}^+$. Since $k + 1 \notin S$, $k + 1 > x$. We have $k \leq x < k + 1 \leq x + 1 < y$, from which $x < k + 1 < y$.

Case 2: $x < 1$, $y > 1$. Then $x < 1 < y$, and we're done.

Case 3: $x < 1$, $0 < y \leq 1$. Then $x < 0 < y$, and we're done.

Case 4: $x < 1$, $-1 < y \leq 0$. Then $x < -1 < y$, and we're done.

Case 5: $x < 1$, $y \leq -1$. Then $-y \geq 1$, and from the first case applied to the pair $-y$, $-x$, we know there is a positive integer z such that $-y < z < -x$. We then have $x < -z < y$, where $-z$ is an integer.

(b) $y - x > 0$, so $1/(y - x) > 0$ as well. Now given any real number u, there is an integer n greater than it. (This is Theorem I.29 in Apostol; it was used to prove the Archimedean property; this was also used in the first solution of (a) above.) Apply this fact in the case $u = 1/(y - x)$; then $n > 1/(y - x) > 0$, and $y - x > 1/n$. Hence $ny - nx > 1$. By part (a) (with x and y replaced by nx and ny), there is an integer z such that $nx < z < ny$, i.e. $x < z/n < y$. Thus z/n is a rational number between x and y.