Math 42 Winter 2006 Practice Exam 2

1. Find the length of the curve with parametric representation

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \frac{\pi}{2}.$$

2. The *gravitational potential* at a point in space is the amount of work it takes (working against gravity) to take an object of mass 1 at that point and move it infinitely far away.

   Suppose an object of mass M is sitting at the origin. The gravitational force it exerts on an object of mass m at a point a distance x away is

$$F = \frac{GMm}{x^2},$$

   where G is a constant. What is the gravitational potential at a point a distance r from the origin?

3. A certain circular sieve with radius 5 cm is more porous at the outside edge than at the center. Suppose that at a distance r cm from the center, an area of 1 cm^2 of the sieve lets through

$$\frac{r^2}{10} + 1$$

   g of sand per second. How much sand does the entire sieve let through in 1 second?

4. Show that

$$y = \frac{\ln t + 3}{t}$$

   is the solution of the initial value problem

$$t^2 y' + ty = 1, \qquad y(1) = 3.$$

5. In a certain country the population grows according to natural growth with relative growth rate $k = \frac{1}{10}$ per year, but crowding also encourages a certain number of people to leave. Suppose that people are leaving the country at a rate $\frac{1}{10}\sqrt{P}$ million people/year, where P is the population in millions.

   (a) Write a differential equation which models the growth of the population.

   (b) What are the equilibrium values of the population?

   (c) What will happen in the long term if there are initially half a million, one million, or four million people?

   (d) Suppose there are initially $P_0 = 4$ million people. Use Euler's method with $h = 5$ to estimate what the population will be in 10 years.

   (e) Use the differential equation to find an exact expression for the population after t years if the initial population is 4 million.

6. The sequence $\{a_n\}$ is defined recursively by

$$a_1 = 4, \quad a_{n+1} = 4 - \frac{3}{a_n}.$$

Show that $\{a_n\}$ is decreasing and $3 \le a_n \le 4$ for all n. Explain why $\{a_n\}$ must be convergent and find its limit.

7. Find the sums of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k-1} - 3}{5^{k+1}}$

8. Determine whether each of the following series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} e^{-1/n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n+3)^{3/2}}$

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^2 + 7}$

9. True / False: (You do not need to justify your answer.)

(a) If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(b) All solutions of $y' = -3 - y^2$ are decreasing functions.

(c) If $\{a_n\}$ and $\{b_n\}$ are divergent then so is $\{a_n b_n\}$.

(d) If $\{a_n\}$ is decreasing and $a_n \ge 2$ for all n, then $\{a_n\}$ is convergent.

(e) The logistic equation has exactly one nonzero equilibrium solution.