MATH 42 Midterm 2 Solutions

Autumn 2006

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Name: ______________________________

Student ID: ___________________________

Name of your TA: ______________________

Instructions. Print your name, student ID number and TA name. Sign below to indicate that you accept **the honor code**. There are **fourteen** pages including this one, and **seven** questions. Before you begin the exam, please make sure that you have all the pages. Read each question carefully and, unless specified otherwise, show all your work and explain your answers. Calculators are not permitted. You have **2 hours** to complete the exam.

Signature: ___________________________

Page	Score	Maximum
3		5+5
4		8+2
5		4+4+4
6		6+4
7		5+5
8		7
9		8
10		4+4
11		4+4
12		6+4
13		5+3
Total		101

Overflow I

1. Consider the following differential equation.

$$\frac{dy}{dx} = \frac{x}{y}$$

(a) Solve the differential equation.

Answer. This differential equation is separable, so

$$\int y dy = \int x dx$$

implies

$$\frac{y^2}{2} = \frac{x^2}{2} + C,$$

or

$$y^2 - x^2 = C,$$

for some real number C.

(b) What is the particular solution that satisfies $y(2) = 1$?

Answer. The initial condition $y(2) = 1$ implies that $1^2 - 2^2 = C$, so the particular solution is

$$x^2 - y^2 = 3.$$

(c) Use Euler's method with step size 1 to estimate $y(5)$ for the solution of the differential equation that satisfies $y(2) = 1$.

Answer. At $(x_1, y_1) = (2, 1)$, we have $y' = 2$, so we follow the line

$$y - 1 = 2(x - 2)$$

to $x_2 = 3$, so $y_2 = 2(3 - 2) + 1 = 3$. Now at $(x_2, y_2) = (3, 3)$, we have $y' = 1$. Follow the line

$$y - 3 = (x - 3)$$

to $x_3 = 4$ with $y_3 = (4 - 3) + 3 = 4$. Now at $(x_3, y_3) = (4, 4)$, we have $y' = 1$. Follow the line

$$y - 4 = x - 4$$

to $x_4 = 5$ with $y_4 = 5$. Thus, Euler's method gives $y(5) \approx 5$.

(d) What is the the error of your approximation?

Answer. From part (b), we see that

$$y(5) = \pm\sqrt{25 - 3} = \pm\sqrt{22}.$$

Thus, the error on the positive value is $|5 - \sqrt{22}|$.

2. Give examples of the following, or explain why no such example exists. Be sure to justify why the example is appropriate.

   (a) A convergent sequence $\{a_n\}_{n=1}^{\infty}$ such that $\sum_{i=1}^{\infty} a_i$ diverges.

Answer. $a_n = 1/n$. The sequence converges to 0, but the harmonic series diverges.

   (b) A series $\sum_{i=1}^{\infty} a_i$ that converges to 9π.

Answer. $a_1 = a_2 = \cdots = a_9 = \pi$, and $a_i = 0$ for $i > 9$.

   (c) A differential equation that has $y = \sin(x^2)$ as a solution.

Answer. The differential equation

$$y' = 2x\cos(x^2)$$

has y as a solution.

3. The following two differential equations model the number of $A(t)$ of ants in a kitchen at time t (measured in weeks).

$$\frac{dA}{dt} = 2A \qquad \text{and} \qquad \frac{dA}{dt} = 20 + A$$

(a) For each, find the particular solution $A(t)$ assuming that the kitchen starts with no ants.

Answer. In the first differential equation, $A(0) = 0$ gives an equilibrium solution, so $A(t) = 0$. The second differential equation is separable, so

$$\int \frac{dA}{20 + A} = \int dt$$

implies

$$\ln |20 + A| = t + C$$

Since $A(0) = 0$, we have

$$C = \ln(20),$$

and by exponentiating,

$$A(t) = 20e^t - 20.$$

(b) Suppose the owners of the kitchen started a campaign to kill the ants. How might this new development change the two differential equations? Give explicit changed differential equations and explain the changes.

Answer. A killing campaign might correspond to either a constant K number of ants killed per week such as

$$\frac{dA}{dt} = 2A - K \qquad \text{and} \qquad \frac{dA}{dt} = 20 + A - K,$$

or killing a certain proportion k of the ants, as in

$$\frac{dA}{dt} = (2 - k)A \qquad \text{and} \qquad \frac{dA}{dt} = 20 + (1 - k)A.$$

4. State whether each of the following statements is TRUE or FALSE. Justify your answers.

   (a) If there exists a constant C such that $a_n \leq C$ for all $n \geq 1$, and

$$a_1 \geq a_2 \geq a_3 \geq \cdots,$$

   then a_n diverges.

Answer. False. The sequence $\{1/n\}_{n=1}^{\infty}$ satisfies all those conditions ($C = 1$), but converges to 0.

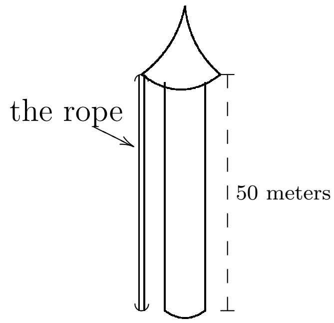
   (b) Since

$$\int_1^{\infty} \sin(\pi x) dx$$

   diverges, by the integral test the series $\sum_{k=1}^{\infty} \sin(\pi k)$ diverges.

Answer. False. The function $\sin(\pi x)$ is not decreasing. In fact, since $\sin(\pi k) = 0$ for all k, the series converges to 0.

5. Mickey Mouse has a 50 meter rope which he accidentally got wet. To dry the rope he hung it from the top of one of his castle towers (which happens to be 50 meters high).



After a couple of hours the rope has not yet dried completely, but since water has accumulated at the bottom we can describe the mass of the rope at a distance y above the ground as

$$e^{-y} \qquad \text{kilograms/meter.}$$

(a) What is the total mass of the rope?

Answer. The mass of a small Δy section of the rope at distance y above the ground has mass approximately

$$e^{-y}\Delta y.$$

Thus, if we add up all the sections and take the limit as the number of sections goes to infinity, we get

$$\text{Mass } = \int_0^{50} e^{-y} dy = -e^{-y}\Big|_0^{50} = 1 - e^{-50}.$$

(b) How much work would it take to lift the entire rope up to the top of the tower? (Recall, if force is constant, then Work = (Force)(Distance); in general, Force = (Mass)(Acceleration), and you can take the acceleration of gravity to be 10 meters/second^2).

Answer. The amount of work to lift a small section of width Δy at height y above the ground to the tower is approximately

$$[\text{Distance}][\text{Force}] = [(50 - y)][10e^{-y}\Delta y].$$

Thus, if we take the limit as the number of sections goes to infinity, we get

$$\text{Work} = \int_0^{50} 10(50 - y)e^{-y}dy = 500\int_0^{50} e^{-y}dy - 10\int_0^{50} ye^{-y}dy$$

and by integrating by parts in the second integral

$$\begin{aligned} &= 500(1 - e^{-50}) - 10(-ye^{-y}\Big|_0^{50} + \int_0^{50} e^{-y}dy \\ &= 500(1 - e^{-50}) - 10(-50e^{-50} + (1 - e^{-50}) \\ &= 490 + 10e^{-50}. \end{aligned}$$

6. Determine which of the following series converge, and IF POSSIBLE determine what they converge to.

   (a) $\sum_{k=1}^{\infty} \cos(\pi k)$

Answer. Since

$$\lim_{k\to\infty} \cos(\pi k) = \lim_{k\to\infty} (-1)^k$$

diverges, by the divergence test, so does the series.

   (b) $\sum_{k=1}^{\infty} \frac{3}{4^k}$

Answer. This is a geometric series

$$\sum_{k=1}^{\infty} \frac{3}{4^k} = \sum_{k=1}^{\infty} \frac{3}{4} \frac{1}{4^{k-1}} = \frac{\frac{3}{4}}{1-\frac{1}{4}} = 1.$$

(c) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$

Answer. Since $\frac{1}{x\ln(x)}$ is positive, continuous and decreasing between $1 \le x$. We can use the integral test. By substituting $u = \ln(x)$, we get

$$\int_2^{\infty} \frac{1}{x\ln(x)}dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x\ln(x)}dx = \lim_{t\to\infty} \int_{\ln(2)}^{\infty} \frac{1}{u}du,$$

which we know diverges. Thus, the series diverges.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{2k+\sqrt{k}}{\ln(k)+k^3}$

Answer. Use the limit comparison test with the sequence $\{1/k^2\}_{k=1}^{\infty}$ to obtain

$$\lim_{k\to\infty} \frac{\frac{2k+\sqrt{k}}{\ln(k)+k^3}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{2k^3+k^{5/2}}{\ln(k)+k^3}.$$

By repeated applications of L'Hospital's rule, we have

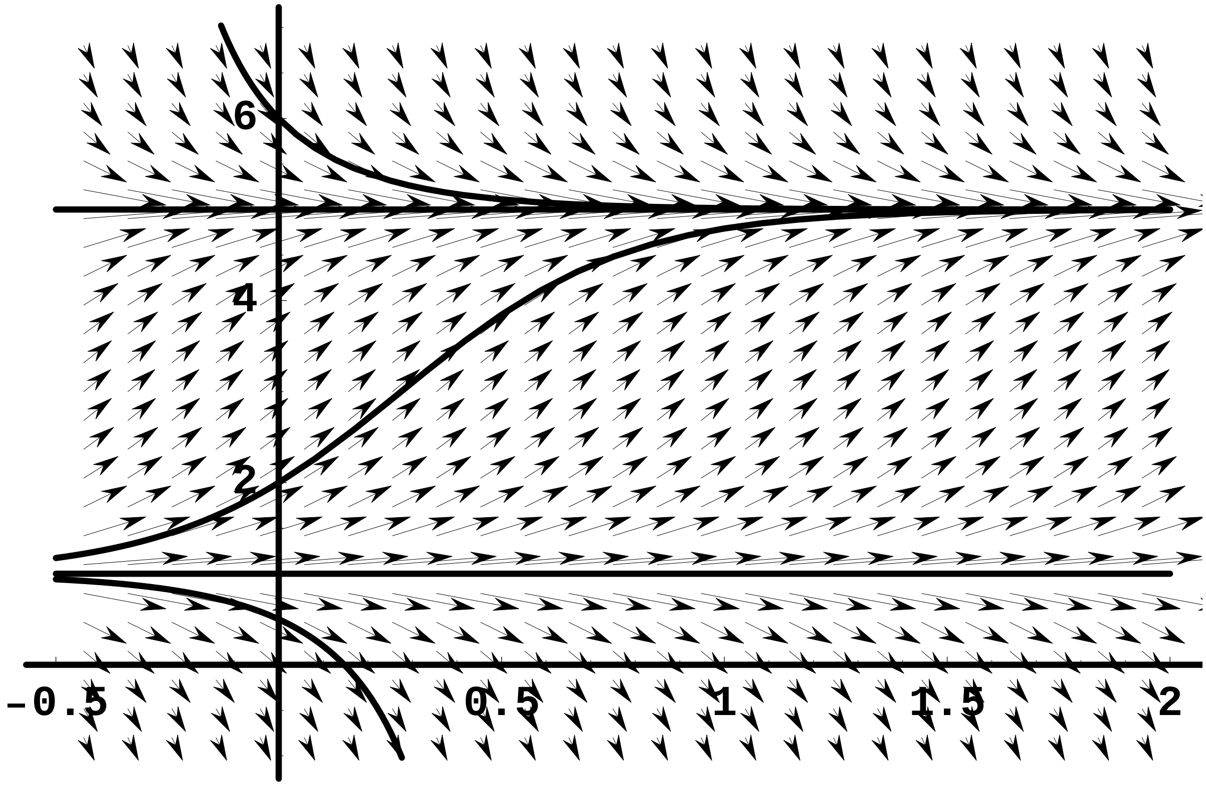
$$\lim_{k\to\infty} \frac{2k^3+k^{5/2}}{\ln(k)+k^3} = 2 > 0.$$

Since $\sum_{k=1}^{\infty} 1/k^2$ converges as a p-series, so does our series.
Alternatively, we could note that

$$\frac{2k+\sqrt{k}}{\ln(k)+k^3} \le \frac{2k+\sqrt{k}}{k^3} = \frac{2}{k^2} + \frac{1}{k^{5/2}}.$$

Since both $\sum_{k=1}^{\infty} \frac{2}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ converge as p-series, by the comparison test, so does our series.

7. (a) The following direction field models a population of fish (in hundreds) in a lake.



i. Sketch the key different longterm behaviors above, and give approximate equilibrium populations.

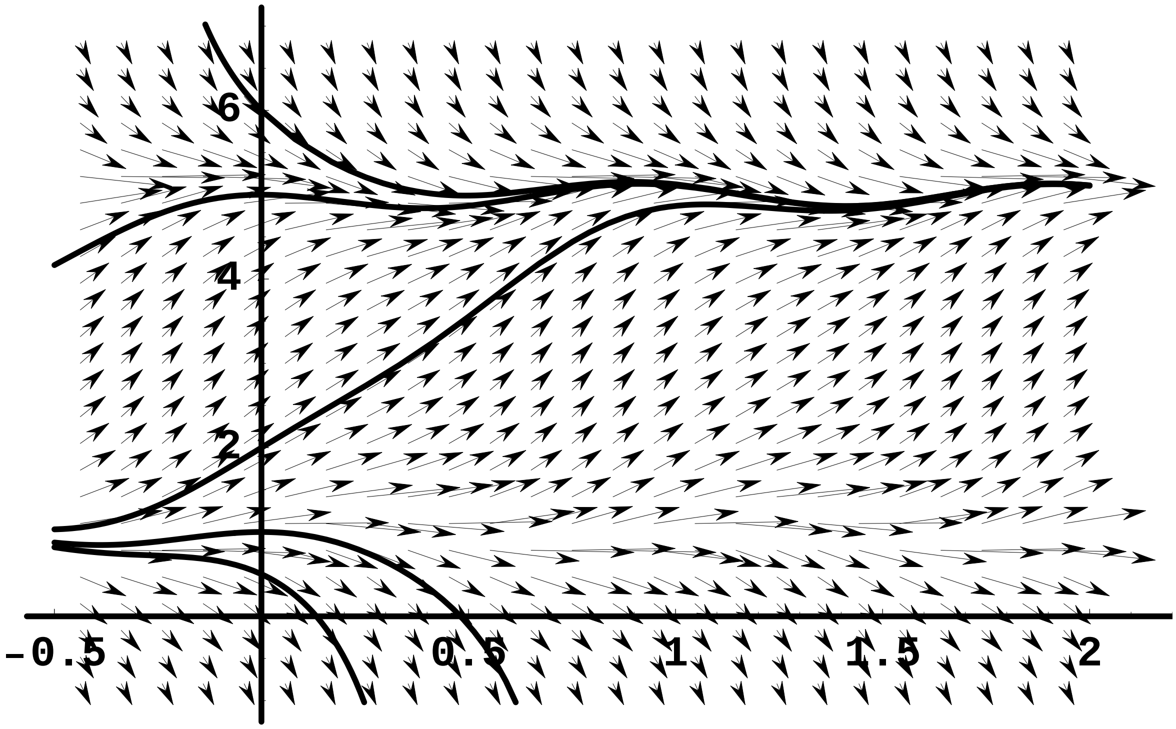
Answer. Equilibrium populations at 100 and 500 fish.

ii. Give an example of a differential equation whose direction field looks something like this one.

Answer.

$$\frac{dP}{dt} = (P - 1)(5 - P)$$

(b) The following direction field comes from a more careful model of the same population of fish (see (a)).



i. Sketch the key different behaviors above. What does this model seem to account for that the other one does not?

Answer. Possible fluctuations in the fish populations: perhaps there is seasonal fishing.

ii. Give an example of a differential equation whose direction field might look something like this one.

Answer.

$$\frac{dP}{dt} = (P-1)(5-P) + \sin(2\pi t).$$

Overflow II