

MATH 42 Midterm 1

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Name: _____

Student ID: _____

Name of your TA: _____

Instructions. Print your name, student ID number and TA name. Sign below to indicate that you accept **the honor code**. There are **eight** two-sided pages including this one, and **twelve** questions. Before you begin the exam, please make sure that you have all the pages. Read each question carefully and, unless specified otherwise, show all your work and explain your answers. Calculators are not permitted. You have **2 hours** to complete the exam.

Signature: _____

Page	Score	Maximum
3		10
4		10
5		6
6		6
7		12
8		10
9		10
10		10
11		10
12		5
13		10
Total		99

Overflow I

1. Compute the following integrals.

$$(a) \int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$$

Answer. By substituting $u = \sqrt{x}$, we have

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx = 2 \int \cos(u) du = 2 \sin(u) + C = 2 \sin(\sqrt{x}) + C,$$

for some C in \mathbb{R} .

$$(b) \int \cos^3(5\theta) d\theta$$

Answer. Use the identity $\cos^2(5\theta) = 1 - \sin^2(5\theta)$, to get

$$\int \cos^3(5\theta) d\theta = \int (1 - \sin^2(5\theta)) \cos(5\theta) d\theta.$$

Substitute $u = \sin(5\theta)$ to get

$$\begin{aligned} \int (1 - \sin^2(5\theta)) \cos(5\theta) d\theta &= \frac{1}{5} \int (1 - u^2) du \\ &= \frac{1}{5} \left(u - \frac{u^3}{3} \right) + C \\ &= \frac{1}{5} \left(\sin(5\theta) - \frac{\sin^3(5\theta)}{3} \right) + C, \end{aligned}$$

for some C in \mathbb{R} .

$$(c) \int \frac{\sqrt{x^2 - 4}}{x} dx$$

Answer. Make the substitution $x = 2 \sec(\theta)$, so $dx = 2 \sec(\theta) \tan(\theta) d\theta$, $\sqrt{x^2 - 4} = 2 \tan(\theta)$, and

$$\int \frac{\sqrt{x^2 - 4}}{x} dx = \int 2 \frac{\sec(\theta) \tan^2(\theta)}{\sec(\theta)} d\theta = 2 \int \tan^2(\theta) d\theta.$$

Note that $\tan^2(\theta) = \sec^2(\theta) - 1$, so

$$2 \int \tan^2(\theta) d\theta = 2 \int \sec^2(\theta) d\theta - 2 \int d\theta = 2 \tan(\theta) - 2\theta + C,$$

for some C in \mathbb{R} . Since $x = 2 \sec(\theta)$, $\tan(\theta) = \sqrt{x^2 - 4}/2$, and

$$2 \tan(\theta) - 2\theta = \sqrt{x^2 - 4} - 2 \sec^{-1}(x/2) + C,$$

for some C in \mathbb{R} .

$$(d) \int_{-1}^1 e^{2x} e^{e^x} dx$$

Answer. Make the substitution $w = e^x$, to get

$$\int_{-1}^1 e^{2x+e^x} dx = \int_{w=1/e}^{w=e} w e^w dw$$

Integrate by parts using $u = w$ and $dv = e^w dw$, to obtain

$$\begin{aligned} \int_{w=1/e}^{w=e} w e^w dw &= w e^w \Big|_{w=1/e}^{w=e} - \int_{w=1/e}^{w=e} e^w dw \\ &= e^{1+e} - e^{1/e-1} - \left(e^w \Big|_{w=1/e}^{w=e} \right) \\ &= e^{1+e} - e^{1/e-1} - e^e + e^{1/e}. \end{aligned}$$

$$(e) \int \frac{x^2 + 5}{x^3 + 2x^2 + x} dx$$

Answer. Note that $x^3 + 2x^2 + x = x(x+1)^2$, so we find A , B , and C such that

$$\frac{x^2 + 5}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

or equivalently,

$$x^2 + 5 = A(x^2 + 2x + 1) + B(x^2 + x) + Cx$$

or equivalently, $A = 5$, $2A + B + C = 0$, $A + B = 1$, so $B = -4$ and $C = -6$, and

$$\begin{aligned} \int \frac{x^2 + 5}{x^3 + 2x^2 + x} dx &= 5 \int \frac{1}{x} dx - 4 \int \frac{1}{x+1} dx - 6 \int \frac{1}{(x+1)^2} dx \\ &= 5 \ln|x| - 4 \ln|x+1| + \frac{6}{x+1} + C, \end{aligned}$$

for some C in \mathbb{R} .

$$(f) \int_0^1 x^2 \ln(x) dx$$

Answer. Note that this integral is improper, so we have

$$\int_0^1 x^2 \ln(x) dx = \lim_{t \rightarrow 0} \int_t^1 x^2 \ln(x) dx.$$

Use integration by parts with $u = \ln(x)$ and $dv = x^2 dx$ to get

$$\begin{aligned} \lim_{t \rightarrow 0} \int_t^1 x^2 \ln(x) dx &= \lim_{t \rightarrow 0} \left. \frac{x^3}{3} \ln(x) \right|_t^1 - \frac{1}{3} \lim_{t \rightarrow 0} \int_t^1 x^2 dx \\ &= \lim_{t \rightarrow 0} \left. \frac{1}{3} \ln(1) - \frac{t^3}{3} \ln(t) - \frac{1}{9} x^3 \right|_t^2 \\ &= \lim_{t \rightarrow 0} -\frac{t^3}{3} \ln(t) - \frac{1}{9}. \end{aligned}$$

In the remaining limit, we get $\infty \cdot 0$, but if we rewrite it

$$\lim_{t \rightarrow 0} -\frac{t^3}{3} \ln(t) = \lim_{t \rightarrow 0} -\frac{\ln(t)}{3t^{-3}},$$

we get ∞/∞ , so we can apply L'Hospital's rule to get

$$\lim_{t \rightarrow 0} -\frac{\ln(t)}{3t^{-3}} = \lim_{t \rightarrow 0} -\frac{t^{-1}}{-9t^{-4}} = \lim_{t \rightarrow 0} -\frac{t^3}{-9} = 0.$$

Thus,

$$\int_0^1 x^2 \ln(x) dx = -1/9.$$

2. Give an example of a function f or explain why no such function exists.

(a) $\int_2^5 f(x)dx$ diverges

Answer. The function

$$f(x) = \frac{1}{x-2}.$$

(b) $\int_0^2 f(x)dx = 9\pi$

Answer. The function

$$f(x) = \frac{9\pi}{2}.$$

(c) $\int f(x)dx = e^{x^2} + C$, for some C in \mathbb{R} .

Answer. The function

$$f(x) = \frac{d}{dx}(e^{x^2} + C) = 2xe^{x^2}.$$

3. TRUE or FALSE. Justify your answer.

(a) If $\int_a^b g(x)dx$ converges and $f(x) \geq g(x) \geq 0$, then $\int_a^b f(x)dx$ converges.

Answer. False, $g(x) = \frac{1}{\sqrt{x}}$ converges from 0 to 1 and $0 \leq \frac{1}{\sqrt{x}} \leq \frac{1}{x}$ on the interval $0 \leq x \leq 1$, but $\int_0^1 \frac{1}{x} dx$ diverges.

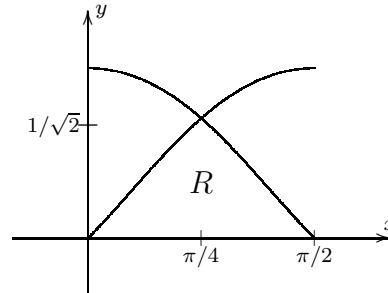
(b) Because $1/x$ is an odd function, $\int_{-1}^1 \frac{1}{x} dx = 0$.

Answer. False, the integral is improper, and $\int_{-1}^0 \frac{1}{x} dx$ diverges, so the whole integral diverges.

4. Let R be the region below both $\cos(x)$ and $\sin(x)$ with $0 \leq x \leq \pi/2$, and above the x -axis.

(a) Sketch R .

Answer.



(b) Write down two different integrals that both give the volume obtained by rotating R around the x -axis. You do not need to evaluate either integral for this part.

Answer. By cylindrical shells, we get

$$2\pi \int_0^{1/\sqrt{2}} y(\arccos(y) - \arcsin(y))dy.$$

By washers, we get

$$\pi \int_0^{\pi/4} \sin^2(x)dx + \pi \int_{\pi/4}^{\pi/2} \cos^2(x)dx.$$

- (c) Find the volume obtained by rotating R around the x -axis (that is, evaluate one of the integrals you found in (b)).

Answer. Using washers, apply the half-angle identities for $\sin^2(x)$ and $\cos^2(x)$ to obtain,

$$\begin{aligned}\pi \int_0^{\pi/4} \sin^2(x) dx + \pi \int_{\pi/4}^{\pi/2} \cos^2(x) dx &= \pi \int_0^{\pi/4} \frac{1 - \cos(2x)}{2} dx + \pi \int_{\pi/4}^{\pi/2} \frac{1 + \cos(2x)}{2} dx \\ &= \frac{\pi}{2} \left(x - \frac{\sin(2x)}{2} \right) \Big|_0^{\pi/4} + \frac{\pi}{2} \left(x + \frac{\sin(2x)}{2} \right) \Big|_{\pi/4}^{\pi/2} \\ &= \frac{\pi}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) + \frac{\pi}{2} \left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2} \right) \\ &= \pi \left(\frac{\pi}{4} - \frac{1}{2} \right).\end{aligned}$$

- (d) Write the length of the boundary of R as a sum of integrals. You do not need to evaluate the integrals.

Answer. By tracing the three parts of the boundary we have,

$$\frac{\pi}{2} + \int_0^{\pi/4} \sqrt{1 + \cos^2(x)} dx + \int_{\pi/4}^{\pi/2} \sqrt{1 + \sin^2(x)} dx.$$

5. (a) Suppose you wish to estimate

$$\int_2^{10} \ln(x) dx$$

using either by the midpoint rule or the trapezoidal rule. Which one would you use if you wanted to be sure to *underestimate* the value of the integral?

Answer. You would want to use the trapezoidal rule. Since $\ln(x)$ is concave down, the trapezoidal rule will add up trapezoids that are completely contained under the curve.

(b) Estimate

$$\int_2^{10} \ln(x) dx,$$

using Simpson's rule by dividing the interval into 4 pieces. You do not need to simplify the expression.

Answer. By Simpson's rule,

$$\int_2^{10} \ln(x) dx \approx \frac{2}{3}(\ln(2) + 4 \ln(4) + 2 \ln(6) + 4 \ln(8) + \ln(10)).$$

(c) What is the largest error you would expect in your answer to (b)?

Answer. Let $f(x) = \ln(x)$. We have

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{6}{x^4},$$

so $|f^{(4)}(x)| \leq \frac{3}{8}$ for all $2 \leq x \leq 10$. Thus, the error will be less than

$$\frac{3 \cdot 8^4}{180 \cdot 4^4}.$$

6. (a) Determine whether or not

$$\int_1^{\infty} \frac{1}{(x+2)(x+3)} dx$$

converges without integrating.

Answer. Note that for $x \geq 1$,

$$0 \leq \frac{1}{(x+2)(x+3)} \leq \frac{1}{x^2},$$

so since $\int_1^{\infty} \frac{1}{x^2} dx$ converges, so does $\int_1^{\infty} \frac{1}{(x+2)(x+3)} dx$.

(b) Find a and b such that

$$\int_a^b \frac{1}{(x+2)(x+3)} dx$$

the integral is still improper, but it does the opposite of your answer for (a).

Answer. Let $a = -2$ and $b = 0$. For $-2 \leq x \leq 0$,

$$\frac{1}{(x+2)(x+3)} \geq \frac{1}{(x+2)3} \geq 0.$$

Since

$$\begin{aligned} \int_{-2}^0 \frac{1}{(x+2)3} dx &= \frac{1}{3} \lim_{t \rightarrow -2} \int_t^0 \frac{1}{x+2} dx \\ &= \frac{1}{3} \lim_{t \rightarrow -2} \ln|x+2| \Big|_t^0 \\ &= \frac{1}{3} \lim_{t \rightarrow -2} \ln(2) - \ln|t+2| \end{aligned}$$

diverges, by the comparison test so does $\int_{-2}^0 \frac{1}{(x+2)(x+3)} dx$.

Overflow II